Laboratorio Interdisciplinario de Estadística Social

Discussion Paper N° 2021|02

A Pseudo-likelihood Approach to the Partial Credit Model: Theory and Applications

Eduardo Rodríguez & Ernesto San Martín

Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Av. Vicuña Mackenna 4860, Macul
lies.mat.uc.cl/

A Pseudo-likelihood Approach to the Partial Credit Model: Theory and Applications

E. RODRÍGUEZ[a] AND E. SAN MARTÍN[b,c]

[a]*Gerencia Corporativa de Datos, CCAF Los Andes, General Calderón 121, Santiago, Chile.*
[b]*Interdisciplinary Laboratory of Social Statistics, Faculty of Mathematics,*
Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Santiago, Chile.
[c]*The Economics School of Louvain, Université catholique de Louvain, Belgium.*

May 24, 2021

Abstract

One of the main properties of Rasch model is the joint representation of both the characteristic of persons and the characteristic of items. This allows to compare two persons by using an item, two items by using a persons, and items and persons. We show that this property is also valid for the Partial Credit Model, emphasizing that this property is established by an identification analysis when the person characteristics are fixed effects. Taking into account that in practice person parameters are estimated in the framework of a random effects model, we propose a simulation study to show the impact of miss-classifications when a random effects model is used rather than a fixed one. The simulations take advantage of an alternative specification of the Partial Credit Model (PCM), called SSB-PCM, where the person parameters are replaced by a pseudo-parameter that are functions of the sum scores. We prove that both specifications provide equivalent estimations and that the SSB-PCM model provide more accurate estimations.

Key words and phrases: Sufficient statistics; Random effects model; Fixed effects model; Sum score based method; Standard setting.

1 Introduction

1.1 Partial Credit Model: Model Specification and Parameter Interpretation

The Partial Credit Model (PCM) is a unidimensional parametric item response theory (IRT) model for the analysis of responses recorded in two or more ordered categories; see Master (1982) and Master and Wright (1997). The ordered categories correspond to several ordered levels of performance on each item and thereby awards partial credit for partial success on items. The usual motive for partial credit scoring is the hope that it will lead to a more precise estimate of a person's ability than a simple pass/fail score. Suppose, for instance, that a person is asked to solve the the following problem (taken from

Master, 1982): $\sqrt{7 \times 5/0.3 - 16}$. The correct answer is obtained after computing $\sqrt{9}$; it corresponds to the highest level of performance. This is getting after mastering two additional levels: the first one corresponds to solve $7 \times 5/0.3$, which is equal to 25; the second one, which supposes to master the first level, corresponds to solve the difference $25 - 16$, which is equal to 9. Thus, the item categories, along with the corresponding partial scores, are the following:

Failed	......	0
$7 \times 5/0.3 = 25$	......	1
$25 - 16 = 9$	......	2
$\sqrt{9} = 3$	......	3

The PCM is specified as follows: consider the responses of n persons to a sequence of k items, $I_1, \ldots, I_i, \ldots, I_k$. Each person may respond to item I_i in $m + 1$ $(m \geq 1)$ *ordered* categories. The response of person v to item I_i will be represented by a selection vector $\boldsymbol{x}'_{vi} = (x_{vi0}, \ldots, x_{vih}, \ldots, x_{vim})$, where $\boldsymbol{x}_{vi}$ is an observation from the random variable $\boldsymbol{X}_{vi}$ defined as follows: $X_{vih} = 1$ if the correct answer is in category h, and $X_{vih} = 0$ otherwise. The model assumes that, for each item, the subject chooses one and only one of the $m + 1$ categories. The probability function of the PCM is, therefore, given by

$$p_{vih} \doteq P(X_{vih} = 1) = \frac{\exp\left(h\theta_v - \sum_{g=0}^{h} \beta_{ig}\right)}{\sum_{z=0}^{m} \exp\left(z\theta_v - \sum_{g=0}^{z} \beta_{ig}\right)}, \quad h = 0, 1, \ldots, m, \tag{1.1}$$

where θ_v is a person parameter representing the ability of person v, whereas β_{ih} correspond to the difficulty of category h of the parameter I_i. The model is completed by assuming that the $\{X_{vih} : v = 1, \ldots, n;\ i = 1, \ldots, k;\ h = 0, \ldots, m\}$ are mutually independent.

Following San Martín et al. (2009), the parameters of interest should be distinguished from the identified parameters. In the case of a PCM, the identified parameters correspond to the probabilities $\{p_{vih} : v = 1, \ldots, n;\ i = 1, \ldots, k;\ h = 0, \ldots, m\}$ because each X_{vih} is distributed according to a Bernoulli distribution of parameter p_{vih} and the mapping $p_{vih} \longmapsto \text{Bern}\,(p_{vih})$ is injective. The parameters of interest are $\{\theta_v, \beta_{ih} : v = 1, \ldots, n;\ i = 1, \ldots, k;\ h = 0, \ldots, m\}$. To identify the parameters of interest, an injective relationship between them and the identified parametrization should be established. The identified quotients $p_{vih}/(p_{vih} + p_{vi,h-1})$ satisfy the following identity (see Master, 1982):

$$\frac{p_{vih}}{p_{vih} + p_{vi,h-1}} = \frac{\exp(\theta_v - \beta_{ih})}{1 - \exp(\theta_v - \beta_{ih})} \qquad h = 1, \ldots, m. \tag{1.2}$$

Consequently, β_{i0} (for $i = 1, \ldots, k$) should be fixed at 0; if not, we would have parameters that are not related to the identified parameters. Furthermore, for each v and i, the differences $\theta_v - \beta_{ih}$ for $h = 1, \ldots, m$ are identified; therefore, a linear restriction is needed to identify (θ_v, β_{ih}) for $h = 1, \ldots, m$. In this paper, the following identification restriction is used:

$$\beta_{i0} = 0 \text{ for } i = 1, , \ldots, k; \quad \beta_{11} = 0. \tag{1.3}$$

Under this restriction, there remains $n + km - 1$ parameters of interest.

The identification restrictions (1.3) also allows us to statistically interpret the parameters of interest; see San Martín et al. (2009). As a matter of fact, the restriction $\beta_{11} = 0$ implies that

$$\theta_v = \ln\left(\frac{p_{v11}}{p_{v10}}\right), \qquad v = 1, \ldots, n. \tag{1.4}$$

Thus, what it is commonly called "ability of person v" corresponds to the logarithm of the ratio between his/her probability to correctly answer the standard item 1 in category 1 and the probability to correctly answer the standard item 1 in category 0. This means that if the individual characteristic $\theta_v > 0$ (respect. < 0), then his/her probability to correctly answer item 1 in category 1 is greater (respect. lesser) than his/her probability to correctly answer item 1 in category 0.

Similarly, for each item i and each category $h \geq 2$, what it is typically called "difficulty of item i in category h" corresponds to

$$\beta_{ih} = \theta_v - \ln\left(\frac{p_{vih}}{p_{vi,h-1}}\right) = \ln\left(\frac{p_{v11}}{p_{v10}} \cdot \frac{p_{vi,h-1}}{p_{vih}}\right). \tag{1.5}$$

Therefore, $\beta_{ih} > \beta_{11} = 0$ has a precise meaning in terms of probability of correctly answer items in specific categories, namely

$$\frac{p_{vi,h-1}}{p_{vih}} > \frac{p_{v10}}{p_{v11}}, \quad h = 2, \ldots, m.$$

Two remarks should be added. First, the statistical interpretation of the parameters θ_v and β_{ih} always involves two categories of an item. Secondly, abilities and difficulties are in the same scale, namely the logarithm of probability ratios; see equations (1.4) and (1.5). Thus, the simultaneous representation ability-difficulty has an explicit statistical meaning: $\beta_{ih} > \theta_v$ means that $p_{vi,h-1} > p_{vih}$, that is, the probability that person v correctly answers item i in category $h - 1$ is greater than his/her probability to correctly answer the same item in category h. In applications this simultaneous representation is crucial, particularly in criterion-referenced measurements; for details, see Berk (1970); Livinston and Zieky (1989); Cizek (2001).

1.2 Estimation procedures in the context of IRT models

IRT models are typically estimated using three different approaches: joint maximum likelihood (JML), conditional maximum likelihood (CML) and marginal maximum likelihood (MML). The first two approaches consider the θ_v's as *parameters*. The JML-approach is the general maximum likelihood method (MLE) applied to the estimation of the parameters of interest. In the context of the Rasch model, it has a number of drawbacks (see Embretson and Reise, 2000, pp. 209-210), one of them being the inconsistency of the difficulty parameters estimates as the number of persons grows (for details, see Andersen, 1980; Ghosh, 1995); this is an example of the famous *incidental parameter problem* (see Neyman and Scott, 1948; Lancaster, 2000). Nevertheless, bias correcting factors have been proposed (see Haberman, 1977; Andersen, 1980), providing empirical evidence that the corrected estimators (denoted as BC-JMLE) work well for a large number of items. The CML-approach uses the fact that the sum score is a sufficient statistics for the ability parameter θ_v (when the difficulty parameters are "fixed"). This leads to factorize the likelihood into two factors: one corresponds to the conditional distribution given the sum score parameterized by the difficulty parameters only; the other one corresponds to the marginal distribution of the sum score parameterized by both the person parameters and the difficulty parameters. The difficulty parameters are estimated maximizing the conditional likelihood given the total score. The abilities can be estimated in a second step from the marginal distribution of the sum score after substituting the difficulty parameters by their estimates. The MML considers the abilities as mutually independent random variables with a common distribution F^{ϕ}, typically a normal one. In this case, model (1.1) is viewed as a conditional distribution given θ_v, and the estimation is focused on the difficulty parameters and the parameter ϕ indexing F. These parameters are consistently estimated provided they are identified by the observations (see, e.g., Kiefer and Wolfowitz, 1956). The abilities are finally estimated using an Empirical Bayes procedure. For details and references, see De Boeck and Wilson (2004) and Baker and Kim (2004).

1.3 Estimation procedures for the PCM

These approaches have also been implemented for the PCM (Master and Wright, 1997; Baker and Kim, 2004). In particular, the JML procedure is widely used in practice in computer programs such as QUEST (Adams and Khoo, 1993) and WINSTEPS (Linacre and Wright, 2000). In this context, Bertoli-Barsotti (2005) provides a necessary and sufficient condition for the existence and uniqueness of the JMLE of both the person parameters and the difficulty parameters. Taking advantage on the structural similarity between the Rasch model and the PCM, the related literature widely accepts (without a formal proof) that the inconsistency of the JMLE for the Rasch model is inherited by the PCM. Accordingly, corrector factors similar to that used for the Rasch model have been proposed for the PCM. Master and Wright (1997) shows, through simulations studies, that such corrector factors work well when compared with both the CML-estimator and the MML-estimator.

1.4 Purpose of this paper

The first concern of this paper is to explain why the JMLE of the PCM does not work correctly. It is actually shown that the JMLE of the PCM is equivalent to the MLE of a misspecified model, which is called *SSB (sum score based)-model*. Broadly speaking, the SSB-model is obtained from the PCM after replacing the person parameter θ_v by a proxy of it based on the sum score of person v (that is, the observed score obtained by person v when answering all the items). The model obtained in this way actually is a conditional model given the sum score; the incorrectness of the SSB-model comes from the fact that it is *explicitly* treated as a marginal model rather than a conditional model. Following the terminology introduced by Besag (1974, 1975), the SSB-model corresponds to a pseudo-likelihood; the MLE obtained from the SSB-model will be denoted as SSBE. It is proved that JMLE $=$ SSBE; the inconsistency of the JMLE is accordingly explained by the fact that it is equal to the MLE of an *incorrect model*, namely the SSB-model. Exact relationships between the estimated standard errors for the PCM and its SSB-version are also obtained. Thus, a complete statistical description of a pseudo-likelihood (in our case, the SSB-model) in terms of the original statistical model (in our case, the PCM) is provided.

Due to the inconsistency problems of the JMLE, PCM are also estimated using a MML-approach. This requires to specify the distribution generating the person abilities F^{φ}, typically a normal distribution $\mathcal{N}(0, \varphi^2)$. The statistical model –obtained after integrating out the person abilities– is indexed by the difficulty parameters β_{ih} and the parameter φ; we call this marginal model *structural PCM*. The person abilities are typically estimated using an Empirical Bayes procedure. In this context, the simultaneous representation of abilities and difficulties mentioned in Section 1.1 is meaningfulness. Taking into account the relevance in applications of such a representation, it should be asked under which conditions the abilities and difficulties estimated in the context of a structural PCM can be considered simultaneously represented. This is the second concern of this paper, which is discussed in Section 5.

This paper is organized as follows: the SSB-formulation of the PCM is developed in Section 2. The equality between the JMLE and the SSBE is proved in Section 3, whereas the relationships between the corresponding standard errors are established in Section 4. In Section 5 a simulation study is conducted in order to compare the classifications of persons if a random effect PCM is used rather than a fixed effect one. Some conclusions are gathered in Section 6.

2 SSB-formulation of the PCM

Let $\boldsymbol{\theta}' = (\theta_1, \dots, \theta_n)$ and $\boldsymbol{\beta}' = (\boldsymbol{\beta_1'}, \boldsymbol{\beta_2'}, \dots, \boldsymbol{\beta_k}')$, where $\boldsymbol{\beta_i'} = (\beta_{i0}, \beta_{i1}, \dots, \beta_{im})$, with $i = 1, \dots, k$. It can easily be verified that the log-likelihood of the PCM is given by

$$l_{\text{JMLE}}(\boldsymbol{\theta}, \boldsymbol{\beta}) = \sum_{v=1}^{n} \theta_v \sum_{h=0}^{m} h x_{v+h} - \sum_{i=1}^{k} \sum_{h=0}^{m} \beta_{ih} \sum_{w=h}^{m} x_{+iw} - G_{\text{JMLE}}(\boldsymbol{\theta}, \boldsymbol{\beta}), \tag{2.1}$$

where $G_{\text{JMLE}}(\boldsymbol{\theta}, \boldsymbol{\beta}) = \sum_{i=1}^{k} \sum_{v=1}^{n} \log \left[\sum_{z=0}^{m} \exp \left(z\theta_v - \sum_{g=0}^{z} \beta_{ig} \right) \right]$. It follows that the sufficient statistics for (θ_v, β_{ih}) is given by

$$\left(\sum_{h=0}^{m} h x_{v+h}, \sum_{w=h}^{m} x_{+iw} \right).$$

From the exponential family theory, it is well known that all persons having the same total score $X_{v++} = \sum_{h=0}^{m} h x_{v+h}$ have the same estimation for their abilities. Thus, a total score $t \in \{0, \dots, mk\}$ can be considered as a proxy of the abilities of persons having a total score $X_{v++} = t$. The SSB-formulation of the PCM is based on this fact. More precisely, the SSB-model corresponding to the PCM is obtained by replacing the θ_v in (1.1) by a $\gamma_t \in \mathbb{R}$ when $v \in I_t = \{v : X_{v++} = t\}$, with $t \in \{0, \dots mk\}$. Thus,

$$p_{vih} \doteq P(X_{vih} = 1 \mid S_v, I_i) = \frac{\exp\left(h\gamma_t - \sum_{g=0}^{h} \beta_{ig}\right)}{\sum_{z=0}^{m} \exp\left(z\gamma_t - \sum_{g=0}^{z} \beta_{ig}\right)}, \quad h = 0, 1, \ldots, m, \quad \text{for all } v \in I_t, \quad (2.2)$$

where γ_t represents a proxy of the ability of a person v who obtained a sum score $X_{v++} = t$. It is still assumed that $\{X_{vih} \ : \ v = 1, \ldots, n, \ \ i = 1, \ldots, k, \ , h = 0, \ldots, m\}$ are mutually independent. This model is called *SSB-model*.

Let N_{tih} be a random variable indicating the number of persons with a sum score equal to t who correctly answered the step h for the item i; let also n_t be the number of persons with a sum score equal to t. Using a sufficiency reduction, model (2.2) can equivalently be rewritten as

$$\boldsymbol{N_{ti}}' \sim \text{Mult}\left(n_t, (p_{i0}^t, \ldots, p_{im}^t)\right), \quad t = 0, \ldots, mk, \ i = 1, \ldots, k, \quad \underset{t,i}{\perp\!\!\!\perp} \boldsymbol{N'_{ti}}, \quad (2.3)$$

where $\boldsymbol{N_{ti}}' = (N_{ti0}, \ldots, N_{tim}), \underset{t,i}{\perp\!\!\!\perp} \boldsymbol{N_{ti}}'$ stands for the mutual independence of $\{\boldsymbol{N_{ti}}' : t = 0, \ldots, mk, \ i = 1, \ldots, k\}$, and for $t = 1, \ldots, mk - 1$, $i = 1, \ldots, k$ and $h = 0, \ldots, m$,

$$p_{ih}^t \doteq p_{vih} \quad \text{when } v \in I_t, \quad (2.4)$$

namely the probability that a person with sum score equal to t correctly answers the step h for the item i. It can be noticed that in (2.3) and (2.4) the total score $t = 0$ and $t = mk$ have been excluded in order to avoid infinite estimates. In the SSB-method, these cases are associated with an absence of randomness, since all the x_{vih} are equal to 0 for $X_{v++} = 0$ and to 1 for $X_{v++} = mk$, respectively; they thus provide no information about the difficulty parameters β_{ih}.

Similarly to the PCM, the identification restrictions of the SSB-model are given by (1.3). Accordingly, the SSB-model involves $t + km - 1$ free parameters. Since $t < n$, the quantity of parameters to be estimated using the SSB-model is smaller than the quantity of parameter of the original PCM. These parameters are estimated using the corresponding log-likelihood, which is given by

$$l_{\text{SSBE}}(\boldsymbol{\gamma}, \boldsymbol{\beta}) = \sum_{t=1}^{mk-1} \gamma_t \sum_{h=0}^{m} h n_{t+h} - \sum_{i=1}^{k} \sum_{h=0}^{m} \beta_{ih} \sum_{w=h}^{m} n_{+iw} - G_{\text{SSBE}}(\boldsymbol{\gamma}, \boldsymbol{\beta}), \quad (2.5)$$

where $\boldsymbol{\gamma}' = (\gamma_1, \ldots, \gamma_t, \ldots, \gamma_{mk-1})$ and $G_{\text{SSBE}}(\boldsymbol{\gamma}, \boldsymbol{\beta}) = \sum_{t=1}^{mk-1} \sum_{i=1}^{k} n_t \log \left[\sum_{z=0}^{m} \exp\left(z\gamma_t - \sum_{g=0}^{z} \beta_{ig}\right) \right]$.
The sufficient statistic for (γ_t, β_{ih}) is, therefore, given by

$$\left(\sum_{h=0}^{m} hn_{t+h}, \sum_{w=h}^{m} n_{+iw}\right).$$

Let us finish this section pointing out that the γ_t's are not properly parameters because they are data dependent through of t, whereas the θ_v's in the PCM are actually incidental parameters. More precisely, the γ_t's depend on the random variable X_{v++}, *but in the SSB-model γ_t's are not longer treated as a random variable, but as a fixed (but unknown) constant*. Not only this fact shows that the SSB-model is a misspecified model, but also that the misspecification arises from ignoring the randomness involved in defining the groups I_t by the sum scores, and the resulting correlation between X_{vih} and the X_{v++}. Using the terminology introduced by Besag (1974, 1975), the SSB-model actually corresponds to a pseudo-likelihood.

3 Equality of the JMLE and the SSBE

In this section we prove that JMLE = SSBE. Both models (1.1) and (2.3) belong to the exponential family; therefore, the corresponding likelihood equations have a unique solution. This uniqueness property means that we need only to verify that the JMLE satisfies the SSB-likelihood equations. The conclusion follows from the uniqueness of the MLE. Let us start by examining the JML-equations:

JML-equations: Since (1.1) belongs to the exponential family, the equations are grouped into two sets: (a.1) $E\left(\sum_{h=0}^{m} hX_{v+h}\right) = \sum_{h=0}^{m} hx_{v+h}$, $v = 1, \ldots, n$; and (b.1) $E\left(\sum_{w=h}^{m} X_{+iw}\right) = \sum_{w=h}^{m} x_{+iw}$, $i = 1, \ldots, k$, $h = 0, \ldots, m$. These equations can respectively be rewritten as

$$\sum_{h=0}^{m} h \sum_{i=1}^{k} p_{vih}(\theta_v, \boldsymbol{\beta}) = \sum_{h=0}^{m} hx_{v+h}, \quad v = 1, \ldots, n. \tag{3.1}$$

$$\sum_{w=h}^{m} \sum_{v=1}^{n} p_{viw}(\theta_v, \beta_{iw}) = \sum_{w=h}^{m} x_{+iw}, \quad i = 1, \ldots, k; \quad h = 0, \ldots, m. \tag{3.2}$$

It follows that $\widehat{\theta}_v$ is the unique solution of $\sum_{h=0}^{m} h \sum_{i=1}^{k} p_{vih}(\eta, \widehat{\boldsymbol{\beta}}) = \sum_{h=0}^{m} hx_{v+h}$ for all $\eta \in \mathbb{R}$. Denoting by $\widetilde{\theta}_t = \widetilde{\theta}_t(\widehat{\boldsymbol{\beta}})$ the unique real number satisfying

$$\sum_{h=0}^{m} h \sum_{i=1}^{k} p_{vih}(\widetilde{\theta}_t(\widehat{\boldsymbol{\beta}}), \widehat{\boldsymbol{\beta}}) = t, \quad t \in \mathcal{T} = \{t | 0 < t < mk\}, \tag{3.3}$$

$\widehat{\theta}_v$ and $\widetilde{\theta}_t$ are linked through the following relation:

$$\widehat{\theta}_v = \widetilde{\theta}_t(\widehat{\boldsymbol{\beta}}) \qquad \text{if } v \in I_t = \left\{ v \middle| \sum_{h=0}^{m} h X_{v+h} = t \right\}. \tag{3.4}$$

Substituting the JMLE in (3.1), we obtain that

$$\sum_{w=h}^{m} \sum_{t\in\mathcal{T}} n_t p_{vih}(\widetilde{\theta}_t(\widehat{\boldsymbol{\beta}}), \widehat{\beta}_{ih}) = \sum_{w=h}^{m} x_{+iw} \quad i = 1, \ldots, k; \quad h = 0, \ldots, m. \tag{3.5}$$

Since the likelihood equations have a unique solution, the JMLE for the PCM is fully characterized by equations (3.3), (3.4) and (3.5).

SSB-likelihood equations: Since (2.4) belongs to the exponential family, the likelihood equations are grouped into two sets: (a.2) $E\left(\sum_{h=0}^{m} h N_{t+h}\right) = \sum_{h=0}^{m} h n_{t+h}$, $t \in \mathcal{T} = \{t | 0 < t < mk\}$; and (b.2) $E\left(\sum_{w=h}^{m} N_{+iw}\right) = \sum_{w=h}^{m} n_{+iw}$, $i = 1, \ldots, k$, $h = 0, \ldots, m$. These equations can respectively be rewritten as follows:

$$n_t \sum_{h=0}^{m} h \sum_{i=1}^{k} p_{ih}^t(\gamma_t, \boldsymbol{\beta}) = \sum_{h=0}^{m} h n_{t+h}, \quad t \in \mathcal{T}. \tag{3.6}$$

$$\sum_{w=h}^{m} \sum_{t\in\mathcal{T}} n_t p_{iw}^t(\gamma_t, \beta_{ih}) = \sum_{w=h}^{m} n_{+iw}, \quad i = 1, \ldots, k, h = 0, \ldots, m. \tag{3.7}$$

Equality between the SSBE and the JMLE: To verify that $\widetilde{\theta}_t(\widehat{\boldsymbol{\beta}})$ and $\widehat{\beta}_{ih}$ are solutions of (3.6) and (3.7) we only need to use two simple facts:

$$\text{(i) } \sum_{h=0}^{m} h n_{t+h} = t n_t; \qquad \text{(ii) } \sum_{w=h}^{m} n_{+ih} = \sum_{w=h}^{m} x_{+ih}.$$

The second fact is true by equations (3.5) and (3.7); the first one comes from the following equality:

$$\sum_{h=0}^{m} h n_{t+h} = \sum_{v\in I_t} \sum_{i=1}^{k} \sum_{h=0}^{m} h x_{vih} = \sum_{v\in I_t} \sum_{h=0}^{m} h x_{v+h} = \sum_{v\in I_t} t = t n_t$$

This ends the proof.

Remark 1 It is important to remark that, when computing the expectations in equations (a.2) and (b.2), it is explicitly forgotten that the total score t is a random variable. It is rather assumed to be a constant; this fact precisely leads to consider the SSB-model as a pseudo-likelihood; furthermore, when fitting the SSB-model, the γ_t 's are treated as constant, and not as random variables.

The SSBE is an unusual instance of pseudo-likelihood estimate, since it agrees exactly with the MLE for the original PCM. This equality implies that the SSBE does not eliminate the inconsistency of the $\widehat{\beta}_{ih}$, even though this cannot now be attributed to the presence of incidental parameters. The source of the inconsistency is that the SSBE is actually derived from the misspecified model (2.2). Reversing the argument, the inconsistency of the $\widehat{\beta}_{ih}$ in the PCM can be explained by the fact that the JMLE coincides with the MLE of a misspecified model.

4 Relationships between the standard errors of the JMLE and the SSBE

In which sense the equality of the point estimates established in Section 3 can be extended to the corresponding (asymptotic) standard errors? This question is motivated by the fact that the Fisher Information Matrices of both the PCM and its SSB-version are related. It is expected, therefore, that their inverses are related and, consequently, the (asymptotic) standard errors too. In this section we establish exact relationships between them using a technique sketched in del Pino et al. (2008). We motivate the kind of relationships we want to establish through an example. A relevant aspect is that these inverse matrices are theoretically obtained without imposing additional hypotheses leading to a simplification of the structure of the Fisher Information Matrices (as done, for instance, by Baker and Kim, 2004). To derive the main results some mathematical machinery is developed, which is explained in Appendix B.

4.1 Relationships between the information matrices

The (asymptotic) standard errors of SSBE and JMLE are the square root diagonal elements of the inverses of the corresponding information matrices. Since (1.1) and (2.2) are generalized linear models, their information matrices coincide with the negative Hessian of the corresponding log-likelihoods (McCullagh and Nelder, 1989). When evaluated at the MLE we denote these matrices by $\mathcal{I}_{\text{JMLE}}$ and $\mathcal{I}_{\text{SSBE}}$, respectively. Under the identification restriction (1.3), the $\mathcal{I}_{\text{JMLE}}$ is given by

$$[\mathcal{I}_{\text{JMLE}}]_{vv} = \sum_{i=1}^{k} \frac{\sum_{z=1}^{m} z^2 \exp\left(z\widehat{\gamma}_t - \sum_{g=1}^{z} \widehat{\beta}_{ig}\right) + \sum_{w=1}^{m-1}\sum_{j=1}^{m-w} w^2 \exp\left((2j+w)\widehat{\gamma}_t - 2\sum_{q=1}^{j}\widehat{\beta}_{iq} - \sum_{r=j+1}^{w+j}\widehat{\beta}_{ir}\right)}{\left(\sum_{z=0}^{m}\exp\left(z\widehat{\gamma}_t - \sum_{g=1}^{z}\widehat{\beta}_{ig}\right)\right)^2}, \quad v \in I_t.$$

$$[\mathcal{I}_{\text{JMLE}}]_{vv'} = 0, \quad v \neq v'.$$

$$[\mathcal{I}_{\text{JMLE}}]_{ih,v} = \frac{\sum_{z=h}^{m} z \exp\left(z\widehat{\gamma}_t - \sum_{g=1}^{z} \widehat{\beta}_{ig}\right) + \sum_{j=1}^{h-1}\sum_{w=h-j}^{m-j} w^2 \exp\left((2j+w)\widehat{\gamma}_t - 2\sum_{q=1}^{j}\widehat{\beta}_{iq} - \sum_{r=j+1}^{w+j}\widehat{\beta}_{ir}\right)}{\left(\sum_{z=0}^{m}\exp\left(z\widehat{\gamma}_t - \sum_{g=1}^{z}\widehat{\beta}_{ig}\right)\right)^2},$$

$$i = 1,\ldots,k; h = 1,\ldots,m, \quad \text{excluding the pair}(i=1, h=1). \tag{4.1}$$

$$[\mathcal{I}_{\text{JMLE}}]_{ih,ih'} = \sum_{t\in\mathcal{T}} n_t \frac{\left(\sum_{w=h'}^{m}\exp\left(w\widehat{\gamma}_t - \sum_{j=1}^{w}\widehat{\beta}_{ij}\right)\right)\left(1 + \sum_{w=1}^{h-1}\exp\left(w\widehat{\gamma}_t - \sum_{j=1}^{w}\widehat{\beta}_{ij}\right)\right)}{\left(\sum_{z=0}^{m}\exp\left(z\widehat{\gamma}_t - \sum_{g=1}^{z}\widehat{\beta}_{ig}\right)\right)^2}, \quad i = 1,\ldots,k; h = 1,\ldots,m; h \le h'.$$

$$[\mathcal{I}_{\text{JMLE}}]_{iz,i'z'} = 0, \quad i \neq i'.$$

Here, we defined $\sum_{w=1}^{0} D_w = 0$, where D_w is an algebraic expression in w. Performing similar computations for $\mathcal{I}_{\text{SSBE}}$ and comparing with (4.1) we obtain the following key relationships:

$$\begin{aligned}
&[\mathcal{I}_{\text{SSBE}}]_{tt} = n_t[\mathcal{I}_{JMLE}]_{vv}, \quad t = 1,\ldots,mk-1; v \in I_t.\\
&[\mathcal{I}_{\text{SSBE}}]_{tt'} = 0 \quad, t \neq t', \quad t = 1,\ldots,mk-1.\\
&[\mathcal{I}_{\text{SSBE}}]_{ih,t} = n_t[\mathcal{I}_{JMLE}]_{ih,v}, \quad i = 1,\ldots,k; h = 1,\ldots,m; t = 1,\ldots,mk-1; v \in I_t.\\
&[\mathcal{I}_{\text{SSBE}}]_{ih,ih'} = [\mathcal{I}_{JMLE}]_{ih,ih'}, \quad i = 1,\ldots,k; h = 1,\ldots,m; h \le h'.\\
&[\mathcal{I}_{\text{SSBE}}]_{ih,i'h'} = 0, \quad i \neq i' \quad, i = 1,\ldots,k.
\end{aligned} \tag{4.2}$$

4.2 Relationships between the standard errors

4.2.1 Illustration

Let us start this section with the following illustration: consider $n = 10$ persons, $k = 5$ items and $h = 2$ categories. Consider the following pattern responses:

$$Y_1 = (0,2,1,1,0), \quad Y_2 = (1,0,1,1,1), \quad Y_3 = (2,2,1,1,1), \quad Y_4 = (2,2,2,0,2), \quad Y_5 = (1,2,1,2,1),$$
$$Y_6 = (1,0,1,0,0), \quad Y_7 = (0,1,1,0,0), \quad Y_8 = (1,0,0,2,0), \quad Y_9 = (1,1,1,0,1), \quad Y_{10} = (1,1,1,1,0).$$

These patterns have not being simulated, but have been arbitrary selected; this does not represent any disadvantage since we are looking for *exact* relationships between the standard errors. As mentioned in Section 2, the SSBE can be efficiently obtained if the data are represented as in Table 1. Using the identification restriction (1.3), the variance-covariance matrix $\mathcal{I}_{\text{SSBE}}^{-1}$ is automatically obtained through the PROC NLIN procedure detailed in Appendix A. Since there are five different total scores, the first five columns (and five rows) of $\mathcal{I}_{\text{SSBE}}^{-1}$ correspond to $\widehat{\gamma}_2, \widehat{\gamma}_3, \widehat{\gamma}_4, \widehat{\gamma}_7, \widehat{\gamma}_8$. Moreover, since we are considering $k = 5$ items with $h = 2$ steps, the second nine columns (and nine rows) correspond to $\widehat{\beta}_{12}, \widehat{\beta}_{21}, \ldots, \widehat{\beta}_{52}$. The corresponding Fisher Information Matrix evaluated at the MLE is the inverse of $\mathcal{I}_{\text{SSBE}}^{-1}$, namely $\mathcal{I}_{\text{SSBE}}$. Both matrices are shown on page 13.

[Table 1 about here.]

Using the Fisher Information Matrix $\mathcal{I}_{\text{SSBE}}$, along with with the information provided by the n_t, namely $n_2 = 2, n_3 = 1, n_4 = 4, n_7 = 2$ and $n_8 = 1$, equality (4.2) leads to obtain the Fisher Information Matrix $\mathcal{I}_{\text{JMLE}}$ corresponding to the JMLE; its inverse $\mathcal{I}_{\text{JMLE}}^{-1}$ corresponds to the variance-covariance matrix of the JMLE of the PCM. Both matrices are shown on page 14. These matrices agree exactly with that obtained after directly fitting the PCM with te SAS procedure PROC NLIN and inverting the estimated covariance matrix. The $n(2k-1)$ person-item combinations and the $n + (2k-1)$ parameters generate a $n(2k-1) \times (n + (2k-1)$ design matrix, which after applying the identifiability restriction is reduced by one column; this is the desing matrix which was used for fitting the model. The first 10 diagonal elements of $\mathcal{I}_{\text{JMLE}}^{-1}$ correspond to the estimated variances of the $\widehat{\theta}_v$'s. There are only 5 different values; the number of repetitions for each of these values is determined by the n_t's, namely $n_2 = 2$, $n_3 = 1$, $n_4 = 4$, $n_7 = 2$ and $n_8 = 1$.

In this example, the following relationships between $\mathcal{I}_{\text{SSBE}}^{-1}$ and $\mathcal{I}_{\text{JMLE}}^{-1}$ can be observed:

1. The block corresponding to the correlations between $\widehat{\beta}_{ih}$ and $\widehat{\gamma}_t$ in $\mathcal{I}_{\text{SSBE}}^{-1}$ is equal to the corresponding block in $\mathcal{I}_{\text{JMLE}}^{-1}$.

2. The variance-covariance matrices of the $\widehat{\beta}_{ih}$'s are equal in both matrices. In particular, s.e.$(\widehat{\beta_{ih}})$ are identical for the SSBE and the JMLE.

3. When $n_t = 1$, the variance of $\widehat{\gamma}_t$ is equal to the variance of $\widehat{\theta}_v$ for $v \in I_t$. Thus $Var(\widehat{\gamma}_3) = Var(\widehat{\theta}_8) = 1.348$ and $Var(\widehat{\gamma}_8) = Var(\widehat{\theta}_4) = 2.277$.

4. When $n_t > 1$, the following inequalities are satisfied:

$$\frac{\text{s.e.}(\widehat{\theta}_v)}{\text{s.e.}(\widehat{\gamma}_2)} = \frac{1.447}{1.002} \leq \sqrt{n_2} = \sqrt{2}, \quad v \in I_2 = \{6, 7\};$$

$$\frac{\text{s.e.}(\widehat{\theta}_v)}{\text{s.e.}(\widehat{\gamma}_4)} = \frac{1.348}{0.914} \leq \sqrt{n_7} = \sqrt{4}, \quad v \in I_4 = \{1, 2, 9, 10\};$$

$$\frac{\text{s.e.}(\widehat{\theta}_v)}{\text{s.e.}(\widehat{\gamma}_7)} = \frac{1.945}{1.577} \leq \sqrt{n_4} = \sqrt{2}, \quad v \in I_7 = \{3, 5\}.$$

4.2.2 Main results

The relationships previously illustrated are *always* valid. As a matter of fact, given that $\mathcal{I}_{\text{JMLE}}$ and $\mathcal{I}_{\text{SSBE}}$ are related through equation (4.2), we can derive a relationship between the standard errors. In order to do so, Appendix B introduces a particular class of partitioned matrices, denoted as $\mathcal{C}(\boldsymbol{n})$, where $\boldsymbol{n}$ is a vector of integers which are used to define the sub-blocks of the corresponding matrix. It can easily be verified that $\mathcal{I}_{\text{JMLE}}$ belongs to this class with $T = m = k - 1$. Using Theorem B.2, the following results are obtained:

$$s.e.(\widehat{\beta}_{ih}) \text{ are identical for the SSBE and the JMLE.} \tag{4.3}$$

$$\left(s.e.(\widehat{\theta}_v)\right)^2 = (s.e.(\widehat{\gamma}_t))^2 + \frac{n_t - 1}{[\mathcal{I}_{\text{SSBE}}]_{tt}} = (s.e.(\widehat{\gamma}_t))^2 + \frac{n_t - 1}{n_t[\mathcal{I}_{\text{JMLE}}]_{vv}}. \tag{4.4}$$

$$\frac{1}{\sqrt{[\mathcal{I}_{\text{JMLE}}]_{vv}} \times s.e.(\widehat{\gamma}_t)} \leq \frac{s.e.(\widehat{\theta}_v)}{s.e.(\widehat{\gamma}_t)} \leq \sqrt{n_t} \quad \forall v \in I_t; \;\; t = 1, \ldots, mk - 1. \tag{4.5}$$

From these relationships it can be concluded that, for $v \in I_t$, $s.e.(\widehat{\theta}_v) \geq s.e.(\widehat{\gamma}_t)$, with equality only attained when there is just one examinee with a sum score equal to t. The upper bound in (4.5) is a useful approximation to $s.e.(\widehat{\theta}_v)$, since it tends to be quite sharp for large-scale test with many items. Moreover, (4.5) implies that $1/\sqrt{[\mathcal{I}_{JMLE}]_{vv}} \leq s.e.(\widehat{\theta}_v)$; here $1/\sqrt{[\mathcal{I}_{JMLE}]_{vv}}$ coincides with the estimated standard error when then parameter estimated are taken as if they were the true values.

For finalizing this subsection, let us illustrate the equality (4.4) with the same example developed in Section 4.2.1. In Table 2 the values of 1.447, 1.348, 1.380, 1.945 and 2.277 for $\left(s.e.(\widehat{\theta}_v)\right)^2$ are identical to the five values on the diagonal of the upper left part of the variance-covariance $\mathcal{I}_{\text{JMLE}}^{-1}$ matrix shown on page 14.

[Table 2 about here.]

$$
\mathcal{I}_{\mathrm{SSBE}}^{-1} = \left(\begin{array}{ccccc|ccccccccc}
1.002 & 0.594 & 0.620 & 0.654 & 0.658 & 0.738 & 0.800 & 0.699 & 0.680 & 0.717 & 0.625 & 0.651 & 0.639 & 0.646 \\
0.594 & 1.348 & 0.699 & 0.765 & 0.774 & 0.783 & 0.747 & 0.774 & 0.770 & 0.838 & 0.765 & 0.767 & 0.760 & 0.764 \\
0.620 & 0.699 & 0.914 & 0.870 & 0.887 & 0.780 & 0.740 & 0.818 & 0.838 & 0.955 & 0.890 & 0.878 & 0.888 & 0.886 \\
0.654 & 0.765 & 0.870 & 1.577 & 1.292 & 0.744 & 0.760 & 0.820 & 0.911 & 1.351 & 1.059 & 1.328 & 1.343 & 1.442 \\
0.658 & 0.774 & 0.887 & 1.292 & 2.277 & 0.752 & 0.762 & 0.812 & 0.896 & 1.429 & 1.064 & 1.512 & 1.413 & 1.649 \\
\hline
0.738 & 0.783 & 0.780 & 0.744 & 0.752 & 1.519 & 0.760 & 0.771 & 0.770 & 0.832 & 0.512 & 0.754 & 0.755 & 0.749 \\
0.800 & 0.747 & 0.740 & 0.760 & 0.762 & 0.760 & 1.996 & 0.754 & 0.752 & 0.828 & 0.746 & 0.720 & 0.754 & 0.758 \\
0.699 & 0.774 & 0.818 & 0.820 & 0.812 & 0.771 & 0.754 & 1.396 & 0.794 & 0.890 & 0.816 & 0.815 & 0.688 & 0.819 \\
0.680 & 0.770 & 0.838 & 0.911 & 0.896 & 0.770 & 0.752 & 0.794 & 1.413 & 0.955 & 0.855 & 0.887 & 0.892 & 0.789 \\
0.717 & 0.838 & 0.955 & 1.351 & 1.429 & 0.832 & 0.828 & 0.890 & 0.955 & 2.204 & 1.072 & 1.284 & 1.274 & 1.367 \\
0.625 & 0.765 & 0.890 & 1.059 & 1.064 & 0.512 & 0.746 & 0.816 & 0.855 & 1.072 & 1.728 & 1.018 & 1.022 & 1.059 \\
0.651 & 0.767 & 0.878 & 1.328 & 1.512 & 0.754 & 0.720 & 0.815 & 0.887 & 1.284 & 1.018 & 2.620 & 1.261 & 1.382 \\
0.639 & 0.760 & 0.888 & 1.343 & 1.413 & 0.755 & 0.754 & 0.688 & 0.892 & 1.274 & 1.022 & 1.261 & 2.248 & 1.354 \\
0.646 & 0.764 & 0.886 & 1.442 & 1.649 & 0.749 & 0.758 & 0.819 & 0.789 & 1.367 & 1.059 & 1.382 & 1.354 & 2.949
\end{array}\right)
$$

$$
\mathcal{I}_{\mathrm{SSBE}} = \left(\begin{array}{rrrrr|rrrrrrrrr}
2.247 & 0 & 0 & 0 & 0 & -0.484 & -0.393 & -0.359 & -0.256 & -0.043 & -0.146 & -0.018 & -0.027 & -0.007 \\
0 & 1.444 & 0 & 0 & 0 & -0.317 & -0.123 & -0.256 & -0.206 & -0.053 & -0.190 & -0.020 & -0.045 & -0.013 \\
0 & 0 & 6.440 & 0 & 0 & -1.128 & -0.296 & -1.107 & -1.029 & -0.390 & -1.175 & -0.148 & -0.401 & -0.132 \\
0 & 0 & 0 & 2.713 & 0 & -0.053 & -0.023 & -0.159 & -0.278 & -0.512 & -0.316 & -0.347 & -0.554 & -0.421 \\
0 & 0 & 0 & 0 & 1.150 & -0.006 & -0.005 & -0.027 & -0.064 & -0.229 & -0.081 & -0.238 & -0.225 & -0.265 \\
\hline
-0.484 & -0.317 & -1.128 & -0.053 & -0.006 & 1.501 & 0 & 0 & 0 & 0 & 0.488 & 0 & 0 & 0 \\
-0.393 & -0.123 & -0.296 & -0.023 & -0.005 & 0 & 0.817 & 0 & 0 & 0 & 0 & 0.023 & 0 & 0 \\
-0.359 & -0.256 & -1.107 & -0.159 & -0.027 & 0 & 0 & 1.687 & 0 & 0 & 0 & 0 & 0.221 & 0 \\
-0.256 & -0.206 & -1.029 & -0.278 & -0.064 & 0 & 0 & 0 & 1.697 & 0 & 0 & 0 & 0 & 0.136 \\
-0.043 & -0.053 & -0.390 & -0.512 & -0.229 & 0 & 0 & 0 & 0 & 1.120 & 0 & 0 & 0 & 0 \\
-0.146 & -0.190 & -1.175 & -0.316 & -0.081 & 0.488 & 0 & 0 & 0 & 0 & 1.419 & 0 & 0 & 0 \\
-0.018 & -0.020 & -0.148 & -0.347 & -0.238 & 0 & 0.023 & 0 & 0 & 0 & 0 & 0.749 & 0 & 0 \\
-0.027 & -0.045 & -0.401 & -0.554 & -0.225 & 0 & 0 & 0.221 & 0 & 0 & 0 & 0 & 1.031 & 0 \\
-0.007 & -0.013 & -0.132 & -0.421 & -0.265 & 0 & 0 & 0 & 0.136 & 0 & 0 & 0 & 0 & 0.701
\end{array}\right)
$$

$$
\mathcal{I}_{\mathrm{JMLE}}^{-1} = \left(\begin{array}{cc|c|cccc|cc|c|ccccccccc}
\mathbf{1.447} & 0.557 & 0.594 & 0.620 & 0.620 & 0.620 & 0.620 & 0.654 & 0.654 & 0.658 & 0.738 & 0.800 & 0.699 & 0.680 & 0.717 & 0.625 & 0.651 & 0.639 & 0.646 \\
0.557 & \mathbf{1.447} & 0.594 & 0.620 & 0.620 & 0.620 & 0.620 & 0.654 & 0.654 & 0.658 & 0.738 & 0.800 & 0.699 & 0.680 & 0.717 & 0.625 & 0.651 & 0.639 & 0.646 \\
\hline
0.594 & 0.594 & \mathbf{1.348} & 0.699 & 0.699 & 0.699 & 0.699 & 0.765 & 0.765 & 0.774 & 0.783 & 0.747 & 0.774 & 0.770 & 0.838 & 0.765 & 0.767 & 0.760 & 0.764 \\
\hline
0.620 & 0.620 & 0.699 & \mathbf{1.380} & 0.759 & 0.759 & 0.759 & 0.870 & 0.870 & 0.887 & 0.780 & 0.740 & 0.818 & 0.838 & 0.955 & 0.890 & 0.878 & 0.888 & 0.886 \\
0.620 & 0.620 & 0.699 & 0.759 & \mathbf{1.380} & 0.759 & 0.759 & 0.870 & 0.870 & 0.887 & 0.780 & 0.740 & 0.818 & 0.838 & 0.955 & 0.890 & 0.878 & 0.888 & 0.886 \\
0.620 & 0.620 & 0.699 & 0.759 & 0.759 & \mathbf{1.380} & 0.759 & 0.870 & 0.870 & 0.887 & 0.780 & 0.740 & 0.818 & 0.838 & 0.955 & 0.890 & 0.878 & 0.888 & 0.886 \\
0.620 & 0.620 & 0.699 & 0.759 & 0.759 & 0.759 & \mathbf{1.380} & 0.870 & 0.870 & 0.887 & 0.780 & 0.740 & 0.818 & 0.838 & 0.955 & 0.890 & 0.878 & 0.888 & 0.886 \\
\hline
0.654 & 0.654 & 0.765 & 0.870 & 0.870 & 0.870 & 0.870 & \mathbf{1.945} & 1.208 & 1.292 & 0.744 & 0.760 & 0.820 & 0.911 & 1.351 & 1.059 & 1.328 & 1.343 & 1.442 \\
0.654 & 0.654 & 0.765 & 0.870 & 0.870 & 0.870 & 0.870 & 1.208 & \mathbf{1.945} & 1.292 & 0.744 & 0.760 & 0.820 & 0.911 & 1.351 & 1.059 & 1.328 & 1.343 & 1.442 \\
\hline
0.658 & 0.658 & 0.774 & 0.887 & 0.887 & 0.887 & 0.887 & 1.292 & 1.292 & \mathbf{2.277} & 0.752 & 0.762 & 0.812 & 0.896 & 1.429 & 1.064 & 1.512 & 1.413 & 1.649 \\
\hline
0.738 & 0.738 & 0.783 & 0.780 & 0.780 & 0.780 & 0.780 & 0.744 & 0.744 & 0.752 & 1.519 & 0.760 & 0.771 & 0.770 & 0.832 & 0.512 & 0.754 & 0.755 & 0.749 \\
0.800 & 0.800 & 0.747 & 0.740 & 0.740 & 0.740 & 0.740 & 0.760 & 0.760 & 0.762 & 0.760 & 1.996 & 0.754 & 0.752 & 0.828 & 0.746 & 0.720 & 0.754 & 0.758 \\
0.699 & 0.699 & 0.774 & 0.818 & 0.818 & 0.818 & 0.818 & 0.820 & 0.820 & 0.812 & 0.771 & 0.754 & 1.396 & 0.794 & 0.890 & 0.816 & 0.815 & 0.688 & 0.819 \\
0.680 & 0.680 & 0.770 & 0.838 & 0.838 & 0.838 & 0.838 & 0.911 & 0.911 & 0.896 & 0.770 & 0.752 & 0.794 & 1.413 & 0.955 & 0.855 & 0.887 & 0.892 & 0.789 \\
0.717 & 0.717 & 0.838 & 0.955 & 0.955 & 0.955 & 0.955 & 1.351 & 1.351 & 1.429 & 0.832 & 0.828 & 0.890 & 0.955 & 2.204 & 1.072 & 1.284 & 1.274 & 1.367 \\
0.625 & 0.625 & 0.765 & 0.890 & 0.890 & 0.890 & 0.890 & 1.059 & 1.059 & 1.064 & 0.512 & 0.746 & 0.816 & 0.855 & 1.072 & 1.728 & 1.018 & 1.022 & 1.059 \\
0.651 & 0.651 & 0.767 & 0.878 & 0.878 & 0.878 & 0.878 & 1.328 & 1.328 & 1.512 & 0.754 & 0.720 & 0.815 & 0.887 & 1.284 & 1.018 & 2.620 & 1.261 & 1.382 \\
0.639 & 0.639 & 0.760 & 0.888 & 0.888 & 0.888 & 0.888 & 1.343 & 1.343 & 1.413 & 0.755 & 0.754 & 0.688 & 0.892 & 1.274 & 1.022 & 1.261 & 2.248 & 1.354 \\
0.646 & 0.646 & 0.764 & 0.886 & 0.886 & 0.886 & 0.886 & 1.442 & 1.442 & 1.649 & 0.749 & 0.758 & 0.819 & 0.789 & 1.367 & 1.059 & 1.382 & 1.354 & 2.949
\end{array}\right)
$$

$$
\mathcal{I}_{\mathrm{JMLE}} = \left(\begin{array}{cc|c|cccc|cc|c|ccccccccc}
\mathbf{1.124} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.242 & -0.196 & -0.180 & -0.128 & -0.022 & -0.073 & -0.009 & -0.014 & -0.003 \\
0 & \mathbf{1.124} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.242 & -0.196 & -0.180 & -0.128 & -0.022 & -0.073 & -0.009 & -0.014 & -0.003 \\
\hline
0 & 0 & \mathbf{1.444} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.317 & -0.123 & -0.256 & -0.206 & -0.053 & -0.190 & -0.020 & -0.045 & -0.013 \\
\hline
0 & 0 & 0 & \mathbf{1.610} & 0 & 0 & 0 & 0 & 0 & 0 & -0.282 & -0.074 & -0.277 & -0.257 & -0.098 & -0.294 & -0.037 & -0.100 & -0.033 \\
0 & 0 & 0 & 0 & \mathbf{1.610} & 0 & 0 & 0 & 0 & 0 & -0.282 & -0.074 & -0.277 & -0.257 & -0.098 & -0.294 & -0.037 & -0.100 & -0.033 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1.610} & 0 & 0 & 0 & 0 & -0.282 & -0.074 & -0.277 & -0.257 & -0.098 & -0.294 & -0.037 & -0.100 & -0.033 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1.610} & 0 & 0 & 0 & -0.282 & -0.074 & -0.277 & -0.257 & -0.098 & -0.294 & -0.037 & -0.100 & -0.033 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1.356} & 0 & 0 & -0.027 & -0.011 & -0.080 & -0.139 & -0.256 & -0.158 & -0.174 & -0.277 & -0.210 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1.356} & 0 & -0.027 & -0.011 & -0.080 & -0.139 & -0.256 & -0.158 & -0.174 & -0.277 & -0.210 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1.150} & -0.006 & -0.005 & -0.027 & -0.064 & -0.229 & -0.081 & -0.238 & -0.225 & -0.265 \\
\hline
-0.242 & -0.242 & -0.317 & -0.282 & -0.282 & -0.282 & -0.282 & -0.027 & -0.027 & -0.006 & 1.501 & 0 & 0 & 0 & 0 & 0.488 & 0 & 0 & 0 \\
-0.196 & -0.196 & -0.123 & -0.074 & -0.074 & -0.074 & -0.074 & -0.011 & -0.011 & -0.005 & 0 & 0.817 & 0 & 0 & 0 & 0 & 0.023 & 0 & 0 \\
-0.180 & -0.180 & -0.256 & -0.277 & -0.277 & -0.277 & -0.277 & -0.080 & -0.080 & -0.027 & 0 & 0 & 1.687 & 0 & 0 & 0 & 0 & 0.221 & 0 \\
-0.128 & -0.128 & -0.206 & -0.257 & -0.257 & -0.257 & -0.257 & -0.139 & -0.139 & -0.064 & 0 & 0 & 0 & 1.697 & 0 & 0 & 0 & 0 & 0.136 \\
-0.022 & -0.022 & -0.053 & -0.098 & -0.098 & -0.098 & -0.098 & -0.256 & -0.256 & -0.229 & 0 & 0 & 0 & 0 & 1.120 & 0 & 0 & 0 & 0 \\
-0.073 & -0.073 & -0.190 & -0.294 & -0.294 & -0.294 & -0.294 & -0.158 & -0.158 & -0.081 & 0.488 & 0 & 0 & 0 & 0 & 1.419 & 0 & 0 & 0 \\
-0.009 & -0.009 & -0.020 & -0.037 & -0.037 & -0.037 & -0.037 & -0.174 & -0.174 & -0.238 & 0 & 0.023 & 0 & 0 & 0 & 0 & 0.749 & 0 & 0 \\
-0.014 & -0.014 & -0.045 & -0.100 & -0.100 & -0.100 & -0.100 & -0.277 & -0.277 & -0.225 & 0 & 0 & 0.221 & 0 & 0 & 0 & 0 & 1.031 & 0 \\
-0.003 & -0.003 & -0.013 & -0.033 & -0.033 & -0.033 & -0.033 & -0.210 & -0.210 & -0.265 & 0 & 0 & 0 & 0.136 & 0 & 0 & 0 & 0 & 0.701
\end{array}\right)
$$

4.2.3 Numerical Illustration of the Main Result

Let us illustrate the theoretical results using a Chilean data set. The data set corresponds to an experimental test applied in 2003. The test was couched in the context of a project leading to implement a new entrance university national test (for details, see Bravo and Manzi, 2004). As a part of this project, a polychotomous test in Mathematics was produced. The test composed of 30 items was applied to 1,090 examinees; each item has 5 alternatives: only one alternative corresponds to 2 points; 7 items contains two alternatives corresponding to 1 point; 23 items contains only one alternative scored with 1 points. Consequently, the items have two steps or categories. The reliability of the test is 0.724.

As it was mentioned in Section 2, examinees with total score equal to 0 or equal to the maximum were excluded, remaining 1,088 students. The estimates and the corresponding estimated standard errors of the difficulty parameters for each item and for each step are gathered in Table 3; they were computed using both a JML-approach and a SSB-approach. As expected, estimates and the estimated standard errors are equal.

[Table 3 about here.]

The estimates for the ability parameters, along with their respective estimated standard errors, can be found in Table 4; they also were computed using both approaches. Several facts can be verified:

1. When $n_t = 1$, as it is the case for $t = 2$, $t = 23$ and $t = 32$, the estimated standard errors are equal.
2. When $n_t > 1$, then $s.e.(\widehat{\theta}_v) > s.e.(\widehat{\gamma}_t)$ when $v \in I_t$, as can be verified in Table 4.
3. The upper bound derived in (4.5) is sharp.
4. The mapping $t \longmapsto \widehat{\gamma}_t$ is increasing.

[Table 4 about here.]

5 Simultaneous representation and the marginal PCM

Although a fixed effects PCM model is quite different from a random effects PCM model, in practice this last version is fitted, but the interpretation of the results id performed under a fixed effects framework. We focus our attention of the following concern: the simultaneous representation of both person characteristics and item characteristics is valid in a fixed-effects framework only. However, if the estimations are obtained under a random effects perspective, the question is to know its impact on the distance between person characteristics and item characteristics. We answer this question in the context of a standard setting procedure.

A simulation study is performed to answer this question. Different probability distributions are used to generate the person characteristcis (also called abilities): ability estimators are obtained using an Empirical Bayes procedure and assuming that the true probability distribution of the random effects is a normal one. It is concluded that the different ability estimators are quite different, whereas the BC-SSB-estimators are almost the same whatever be the true distribution generating the abilities. We argue that when any information on the probability distribution generating the random effect is available, the BC-SSBE of the abilities is more precise than the corresponding Empirical Bayes Estimators (EBE), since the variability of the different EBE is avoided.

5.1 Standard setting and criterion-referenced measurement

The SSB-estimators could also be useful when respondents are categorized in different levels of proficiency with respect to a particular criterion or set of criteria. The categories are most often dichotomous, like in classifying masters and non-masters, but they can also be polytomous, like in grading the performance on an exam. Most standard setting methods rely on a continuum view of mastery (Meskauskas, 1976): mastering a trait or educational objective is conceived as a gradual process. A cutoff point on the continuum indicates the point of sufficient proficiency to be classified as a master. Judgments of experts are used to determine the cutoff points on the observed test score scale. In an examinee-centered method, judges classify respondents into masters, non-masters and/or borderline cases. The cutoff score is set by determining the point on the observed test score scale that is most consistent with these classifications. In a test-centered method, like for example the method of Angoff (1971), judges review the items in the test, by rating the probability of success that is expected from masters of the domain. The cutoff on the test score scale is set at the sum of the expected performances on the items of the test.

The continuum view of mastery implies that the acquisition of the underlying trait or ability measured by the criterion-referenced test consists in a progression along a continuum. Hence, when a criterion-referenced test is administered to a heterogeneous sample of respondents, one can expect that the respondents are ordered along one dimension. This assumption is implicit in the use of the observed test score as an indication of the level of proficiency on the measured criterion. However, it can also be inferred from the continuum view of mastery that the items of a criterion-referenced test can in principle be ordered along the same continuum as a function of their difficulty. This idea of ordering the items along the same continuum as the persons seems to be implicit in Glaser (1963)'s original paper on criterion-referenced measurement: "the standard against which a student's performance is compared ... is the behavior which defines each point along the achievement continuum" (Glaser, 1963, p. 519). By consequence, "along such a continuum of attainment, a student's score on a criterion-reference measure provides explicit information as to what the individual can or cannot do" Glaser (1963, p. 519-520).

The assumption of ordering both respondents and items on the same continuum naturally leads to unidimensional IRT models as the adequate formalization of the continuum view of mastery. In these models, persons and items are positioned on the same continuum on the basis of their ability and difficulty, respectively, so that their positions can be compared directly; for details, see Janssen et al. (2000). However, this procedure is valid in an IRT model where the abilities are viewed as unknown parameters.

As a matter of fact, for the PCM, equality (1.2) implies that

$$\ln\left(\frac{p_{vih}}{p_{vi,h-1}}\right) = \theta_v - \beta_{ih} \quad v = 1, \ldots, n;\ i = 1, \ldots, k;\ h = 0, \ldots, m.$$

This means

Although IRT models have been used for criterion-referenced measurement, they have not been used as part of a standard setting procedure, which would require a method to define a cutoff point on the latent scale. In general, given that a criterion-referenced measure for defining mastery in a certain domain conforms to a unidimensional IRT model, one could proceed in two ways to set a standard, just like in het existing methods using the observed test score scale. In an examinee-centered approach, one could determine the cutoff as a function of the performance of a selected group of respondents. In a test-centered approach, one could determine the cutoff as a function of the expected average performance of a master on the items. In the hierarchical IRT model, the latter approach is followed, as will be explained below.

Suppose we apply a polychotomous test to a population of examinees, and we are interesting in selecting a group of them. For instance, we choose examinees whose is higher than a pre-fixed cut score. Therefore, the attention is focused on the ranking of the examinees (based on their estimated abilities) rather than on the values of such estimates. A simulation study (see section **??** for details) shows that such a ranking is invariant whatever be the distribution of the random effects. More precisely, we empirically show that if the abilities are generated from different probability distributions and we estimate them assuming that the true probability distribution is a $N(0, \sigma^2)$, the rankings of the examinees are always the same. This phenomenon is due to the fact that the $\widehat{\theta}_i$'s (estimated from a Empirical Bayes procedure) always have (in the simulations study) a monotonic relationship with the sum score. Taking into account that the SSB-estimates of the abilities are an increasing function of the row score, the ranking induced by these estimates is, therefore, equivalent to the ranking induced by the Empirical Bayes estimates, which in turn is independent of the true distribution generating the individual abilities. The SSB-procedure has an extra advantage, namely it can be applied to a very large data sets (since the design matrix only depends on the number of items, not on the number of examinees), it is fast and only use a simple SAS code. Therefore, if we are interesting in ranking examinees based on a polychotomous test, SSB-procedure seems to be an efficient option.

In what follows, we provide details about the rationale of the simulation study; thereafter, we comment the results.

5.2 Design of the Simulation Study

A simulation study consisting of the following four parts was performed:

(a) A test composed of $k = 30$ items, each of them with $h = 2$ categories, was considered. The "true" difficulties correspond to the estimates of the item parameters obtained with SSBE method with a sample of 50 students taken from the SIES test; see Table 5.

[Table 5 about here.]

(b) $n = 200$ individual abilities were generated from specific distributions, which were different from a normal distribution $\mathcal{N}(\mu, \sigma^2)$.

(c) From (a) and (b), the corresponding patterns responses were generated using the PCM.

(d) Using the patterns responses obtained in (c) both items parameters and abilities were estimated by means of two estimation procedures: (i) NLMIXED (from SAS) with a normal distribution $N(0, \sigma^2)$ as the distribution of the abilities; (ii) SSB-method as implemented in SAS. Let us remark that the estimations of the abilities obtained with the NLMIXED are computed with an Empirical Bayes procedure.

It is relevant to make explicit the random-effects specification of the PCM. Let

$$\beta_i = (\beta_{i0}, \ldots, \beta_{im})' \quad \text{for} \quad i = 1, \ldots, k$$
$$X_v = (X_{v1}, \ldots, X_{vk})' \quad \text{for} \quad v = 1, \ldots, n$$

where $X_{vi} = (X_{vi0}, \ldots, X_{vim})'$. The hypotheses underlying the PCM can be written as follows:

(i) X_{vi} depends on (θ_v, β_i) according to equation (1.1).

(ii) For each person v, his/her responses $\{X_{vi}, \ldots, X_{vk}\}$ are mutually independent conditionally on $(\theta_v, \beta_1, \ldots, \beta_k)$. This is the so-called Axiom of Local Independence. It means that the process generating X_{vi} conditionally on $(X_{v1}, \ldots, X_{v,i-1}, X_{v,i+1}, \ldots, X_{vk}, \theta_v, \beta_1, \ldots \beta_k)$ only depends on $(\theta_v, \beta_1, \ldots, \beta_k)$; that is, the responses of person v to the items only depends on his/her ability θ_v and on the difficulty parameters. In the classical literature, this was called the Hypothesis of the Common Cause; see Laplace (1820) and Reichenbach (1956). For details on the Axiom of Local Independence, see Lazarsfeld (1950).

(iii) $X_1, \ldots, X_n$ are mutually independent given $\theta_1, \ldots, \theta_v, \beta_1, \ldots, \beta_k$. This means that once the abilities and difficulties are known, the responses of the examinees are mutually independent.

(iv) $(\theta_i \mid \varphi) \overset{iid}{\sim} F^{\varphi}$, where F is a known probability distribution parameterized with $\varphi \in \Phi$.

(v) $\varphi \perp\!\!\!\perp (\beta_1, \ldots, \beta_k)$.

In order to grasp the meaning of condition (v), two comments should be added:

1. From (i) and (ii), it follows that
$$X_v \perp\!\!\!\perp \varphi \mid \theta_v, \beta_1, \dots, \beta_k \tag{5.6}$$
Similarly, from (iv) it follows that
$$\theta_v \perp\!\!\!\perp \beta_1, \dots, \beta_k \mid \varphi \tag{5.7}$$
Condition (5.6) means that the parameters $(\beta_1, \dots, \beta_k)$ are sufficient to describe the conditional process X_v given θ_v. Condition (5.7) means that φ is sufficient to describe the marginal process generating θ_v. In this context , $\varphi \perp\!\!\!\perp \beta_1, \dots, \beta_k$ defines a cut(see Barndorff-Nielsen (1978); Florens et al. (1990a), Chapter 3); this means that the conditional process and the marginal process are cutted in the sense that the parameters describing both the conditional and the marginal processes are not functionally related.

2. In condition (v) the prior distributions are left unspecified. The only relevant aspect here is the structure of the specification, and not the computation of posterior distribution. Therefore, this way of specifying specify the model is also valid from a sampling-theory framework. In fact, condition (v) could be replaced by a variation-free property, but in most of the cases, it is difficult to characterized it; for details, see Engle et al. (1983).
The hypotheses above-mentioned imply that $X_1, \dots, X_n$ are mutually independent with a common distribution given by
$$P[X_v = x_v \mid \beta_1, \dots, \beta_k, \varphi] = \int_{\mathbb{R}} \prod_{i=i} P[X_v = x_v \mid \beta_i, \theta] F^{\varphi}(d\theta)] \tag{5.8}$$
where $P[X_v = x_v \mid \beta_i, \theta]$ should be computed using (1.1). For a proof, see Mouchart and San Martin (2003).
The SAS-procedure NLMIXED use (5.8) to estimated the parameters of interest. In our case , we use F^{φ} as a $N(0, \sigma^2)$. To fit the model, any identification restriction is needed. In fact, it can be proved that $(\beta_1, \dots, \beta_k, \sigma^2)$ is identified by X_1; it es enough to apply the arguments developed by San Martín and Rolin (2007) to equation (1.2) $(\theta_v \mid \sigma^2) \sim N(0, \sigma^2)$ in order to identify (β_{ih}, σ^2).

Using this simulation design, five studies were performed, each with a different distribution for the abilities or random effects (see step (b) above):

Normal distribution	$\mathcal{N}(-0.2, 1)$
Skew Normal distribution, skewed to the right	$\mathcal{SN}(\lambda = 2, \mu = -0.8, \omega = 1.3)$
Skew Normal distribution, skewed to the left	$\mathcal{SN}(\lambda = 5, \mu = 1.4, \omega = 1.2)$
Mixture of Normal Distributions with equal weights	$0.5\mathcal{N}(-2.7, 1.3) + 0.5\mathcal{N}(1.5, 1.2)$
Mixture of Normal Distributions with different weights	$0.7\mathcal{N}(-1.9, 1.8) + 0.3\mathcal{N}(2.5, 1.3)$.

For a skew normal distribution $\mathcal{SN}(\lambda, \mu, \omega)$, λ corresponds to the skew parameter, μ to the location parameter, and ω to the scale parameter. The parameters of these distributions were chosen in such a way that the true-β values -see step (a) above- lies where the mass of the distribution is 0.9 ($\pm$). The individual abilities were generated as follows:

(a) For mixtures of normal distribution $pN(\mu_1, \sigma_1^2) + (1-p)N(\mu_2, \sigma_2^2)$, first a $u \sim U(0,1)$ is generated. Secondly, if $u < p$, the θ is generated from a $N(\mu_1, \sigma_1^2)$; if not, θ is generated from a $N(\mu_2, \sigma_2^2)$.

(b) For the skew normal distribution, it was essentially used the following stochastic representation: if $X \sim \mathcal{SN}(\lambda)$, then

$$X \stackrel{d}{=} \frac{\lambda}{\sqrt{1+\lambda}}|U| + \frac{1}{\sqrt{1+\lambda}}V$$

where U and V are independent $\mathcal{N}(0,1)$; see Henze (1986). Thus, a random variable $Y \sim \mathcal{SN}(\lambda, \mu, \omega)$ was simulated using the following stochastic representation: $Y \stackrel{d}{=} \mu + \omega X$.

For the mixture normal distributions, steps (c) and (d) were replicated $N = 10$ times, whereas for the other three distributions $N = 50$ replications were performed. The focus of the simulation studies is to compare estimates of the difficulties and the individual abilities by using the SSB-method (with bias-corrected factors) and the NLMIXED procedure. Consequently, we are interested in comparing these estimations for each replication, and not to compare Monte-Carlo estimators. Furthermore, each replication does not show the same total scores and, accordingly, if the abilities estimations were computed using the Monte-Carlo procedure, then each would be based on a different number of replications. Finally, let us mention that each NLMIXED procedure, including the Empirical Bayes computation, took 3 hours in a PC Intel(R) Xeon(TM) CPU 3.00 GHz 1.00 of RAM.

5.3 Discussion of the results I: difficulty parameters

From the simulation study, two types of findings need to be discussed: that related to the β-estimates, and that related to the θ-estimates. For the three different distributions of the random effects, the β-estimates obtained by both the MML method and the SSB method are practically invariant between them, and also very near to the true difficulties. Taking into account that the β-estimates are monotonic functions of the empirical difficulties, the correlation between $\widehat{\beta}_{SSBE}$ and $\widehat{\beta}_{MMLE}$ is almost 1. However, this is not enough to empirically prove an almost perfect agreement between $\widehat{\beta}_{SSBE}$ and $\widehat{\beta}_{MMLE}$.

A way to compare $\widehat{\beta}_{SSBE}$ and $\widehat{\beta}_{MMLE}$ is through the distances between difficulties, namely $\widehat{\beta}_{SSBE,ih} - \widehat{\beta}_{SSBE,i-1,h}$ and $\widehat{\beta}_{MMLE,ih} - \widehat{\beta}_{MMLE,i-1,h}$ (for example, see figure 1 and 2 for both methods). This is the right way to make a comparison because both difficulties and abilities are represented in an interval scale; the role of the identification restriction is to fix the 0 of the scale.

[Figure 1 about here.]

[Figure 2 about here.]

The invariance of the estimations of the difficulty parameters with respect to the misspecification of the distribution of the individual abilities is due to the sufficiency of total score with respect to φ. In fact, from the exponential family theory it follows that X_{v++} is sufficient for θ_v (when the β's are fixed). This can be written as

$$X_v \perp\!\!\!\perp \theta_v \mid X_{v++}, \beta_1, \ldots, \beta_k \tag{5.9}$$

On the other hand

$$X_v \perp\!\!\!\perp \varphi \mid \theta_v, \beta_1, \ldots, \beta_k; \tag{5.10}$$

but X_{v++} is a function of X_v. It follows that

$$(X_v, X_{v++}) \perp\!\!\!\perp \varphi \mid \theta_v, \beta_1, \ldots, \beta_k, \tag{5.11}$$

which in turn implies that

$$X_v \perp\!\!\!\perp \varphi \mid \theta_v, X_{v++}, \beta_1, \ldots, \beta_k, \tag{5.12}$$

5.9 and 5.12 are equivalent to

$$X_v \perp\!\!\!\perp \varphi, \theta_v \mid X_{v++}, \beta_1, \ldots, \beta_k, \tag{5.13}$$

which implies that

$$X_v \perp\!\!\!\perp \varphi \mid X_{v++}, \beta_1, \ldots, \beta_k. \tag{5.14}$$

That is, X_{v++} is a sufficient statistics for φ given $\beta_1, \ldots, \beta_k$. Let us remark that classical sufficiency and Bayesian sufficiency are equivalent for all prior distribution; for details, see Florens et al. (1990b).

Therefore, the likelihood (for one person) obtained after integrating out the random effect θ_v can be factorized as

$$\begin{aligned} P[X_v = x_v \mid \varphi, \beta_1, \ldots, \beta_k] = {} & P[X_v = x_v \mid X_{v++}, \beta_1, \ldots, \beta_k] \times \\ & \times \int P[X_{v++} = t \mid \theta, \beta_1, \ldots, \beta_k] \cdot F^{\varphi}(d\theta) \end{aligned}$$

Thus, the conditional likelihood $P[X_v = x_v \mid X_{v++,\beta_1,\ldots,\beta_k}]$ only provides information for the β's, and consequently the misspecification of F^{φ} does not affect the estimation of them. Let us remark that, in the statistical literature, it is known (also through simulations) that the estimation of the fixed effects (in our case, the difficulty parameters) are robust with respect to the miss-specification of the distribution of the random effects(in our case, the individual difficulties); see Agresti et al. (2004), Heagerty and Kurland (2001), Verbeke and Lesaffre (1996).

Taking into account these considerations, the BC-SSBE of the difficulty parameters can in practice be used to describe the structure of the test. The computational efficiency of the BC-SSBE is an advantage for this kind of descriptions.

5.4 Discussion of the results II: person parameters

The concern of this section is to compare classifications of persons using both their $\widehat{\theta}$-estimates and their θ-true abilities. To do that, we simulate a standard setting procedure. The basic ideas are the following:

1. A set of judges receive the items ordered by difficulty.

2. They are asked to put a mark on the item they consider a minimal competent student should answer.

3. It is assumed in the literature that the minimal competent student answer this item with a probability of 0.67 or 0.70 or 0.85. Thus, judgement of each judge can be transformed into the ability of a minimal competent student by solving the following equation:

$$\frac{\exp\left(2\theta_v - \sum_{g=0}^{2}\beta_{ig}\right)}{\sum_{z=0}^{2}\exp\left(z\theta_v - \sum_{g=0}^{z}\beta_{ig}\right)} \leq 0.7 \tag{5.15}$$

In this application we suppose that all the judges arrive at the same $\widehat{\theta}_{MCS}$. In practice, this value is obtained as the median of different $\widehat{\theta}_{MCS}$, or other tendency measure. Moreover, we compute the $\widehat{\theta}_{MCS}$ assuming that the probability (5.15) is at most equal to 0.70. Once the $\widehat{\theta}_{MCS}$ is obtained, we compute the classifications induced by $\widehat{\theta}_{SSBE}$ and $\widehat{\theta}_{MML}$ with the true classifications (based on the true abilities). If the estimated ability $\widehat{\theta}_{MC}$ is such that $\widehat{\theta}_{MC} \leq 0.7$, we say that the corresponding examinee FAILS the selection test; in other case, we say that he/she PASSES. Similarly, for the true-ability.

Tables 6 , 7 and 8 show the average of the simulations previously described. The reported results were computed using two types of true distributions: mixture of normal distributions and a normal distribution. These tables should be read as follows: in Table 6, consider the first row. The 88.53% of the examinees who failed (taking into account their true scores) were classified as FAILED EXAMINEES taking into account their abilities estimated MML-procedure; the 11.47% corresponds, therefore, to the proportion of failed examinees (w.r.t. their true scores) who were classified as PASSED EXAMINEES when they are classified using their abilities estimated with the MML-procedure. Similarly, for the abilities estimated with the SSB-procedure. Tables 6 , 7 and 8 reported classifications of examinees with respect to two items: item 18 and item 3. The difficulty of these items approximatively corresponds to the solution of equation (5.15) for 0.67 and 0.70, respectively.

Tables 6 and 7 report the results when the true distribution generating the abilities are estimated assuming that such a distribution is a normal one. It can be viewed that, for the item 18, the classifications performed with the SSB-procedure are better than the corresponding classifications performed with the MML-procedure. However, for the item 3, the classifications performed with the MML-procedure is better than the corresponding classifications performed with the SSB-procedure. More precisely, the SSB-procedure is better than the MML-procedure when classifying failed examinees, whereas the MML-procedure is better than the SSB-procedure when classifying passed examinees. Table 8 shows the results when the true distribution is a normal one. The conclusions are similar to the previous ones. These results show, therefore, that the estimators of the random effects when compared with the SSB-estimators are in general bad when compared with the true values.

[Table 6 about here.]

[Table 7 about here.]

[Table 8 about here.]

6 Concluding remarks

This paper deals with the use and limitations of a pseudo-likelihood estimation method which can be employed as an alternative for a common estimation method used for the JML formulation of the Partial Credit Model. The first result is that the alternative method, the sum score based estimation (SSBE), provides point estimates which are proven to be identical to those of the JMLE. The equality of the point estimates allows the JMLE to be interpreted as a pseudo-likelihood estimate, and this offers some insight in the features of the JMLE for the PCM model.

The second result is that the standard errors for the JMLE and the SSBE are equal for the difficulties, but not for the abilities. The equality for the difficulties is rather surprising, since in pseudo-likelihood estimation the standard errors typically do not agree with those of the MLE for the true model. As far as the abilities are concerned, an exact formula relating the standard error of the JMLE to that of the SSBE is provided. This is supplemented by upper and lower bounds on the ratio between the two standard errors, one of which is quite sharp.

In order to obtain the results for the standard errors, a special class of patterned partitioned matrices has been defined and it has been shown how to obtain their inverses efficiently, something that may be useful beyond its application to this paper. Moreover, the Fisher Information Matrix evaluated at the JMLE can be exactly recovered from the estimated covariance matrix of the SSBE.

The relationships established in this paper are not only of theoretical interest, but they have also a practical value. They imply that standard software for the estimation of generalized linear models (GLM) can be used for the joint maximum likelihood estimation without a complicated set up to estimate an ability parameter for each person. The standard errors and information matrix for the JMLE estimates of the individual abilities can be obtained through rather simple equations starting from the SSBE results. In particular, the SSB-estimators helped us to compare the estimations of the difficulty parameters with that obtained when the abilities are interpreted as a latent variable and their distribution is misspecified. Through simulations, we prove that these estimators are quite similar. Using the jargon of generalized linear mixed models, we would say that the estimation of the fixed effects is robust with respect to the misspecification of the distribution of the random effects. We also offer a theoretical justification of this fact. A similar comparison was performed between the MLE of the abilities with the predictions of them using an Empirical Bayes procedure. The simulation results show that these estimators are not robust with respect to the misspecification of the distribution of the random effects and, consequently, a standard setting should be based on the fixed effects PCM model, not the random effects version. This is not just a matter of convenience, but a conceptual one: the simultaneous representation of persons and items is only valid in the faxed-effect framework of the model.

[Table 9 about here.]

[Table 10 about here.]

(*) At least, one of the estimated standard errors is greater than 15.00. This value is the minimum of the large estimated standard errors obtained in same simulations. The maximum is VER EL VALOR , the maximum quantity of large estimated standard errors in a simulation is 10, and the average is 3 large values.

[Table 11 about here.]

[Table 12 about here.]

[Table 13 about here.]

A SAS code for fitting the SSB-version of the PCM

Let k be total number of items, n be the total number of persons, and h be the total number of steps or categories. The original data set need to be r educed to a design matrix of the following form:

$$
\text{data file} = \left(\begin{array}{cccc|c|ccc}
1 & & & & & n_{111} & n_{112} & n_1 \\
1 & & & & I_k & n_{121} & n_{122} & n_1 \\
\vdots & & & & & & \vdots & \\
1 & & & & & n_{1k1} & n_{1k2} & n_1 \\
\hline
 & 1 & & & & n_{211} & n_{212} & n_2 \\
 & 1 & & & I_k & n_{221} & n_{222} & n_2 \\
 & \vdots & & & & & \vdots & \\
 & 1 & & & & n_{2k1} & n_{2k2} & n_2 \\
\hline
 & & \ddots & & \vdots & & \vdots & \\
\hline
 & & & 1 & & n_{t11} & n_{t12} & n_t \\
 & & & 1 & I_k & n_{t21} & n_{t22} & n_t \\
 & & & \vdots & & & \vdots & \\
 & & & 1 & & n_{tk1} & n_{tk2} & n_t
\end{array}\right)
$$

The SAS-procedure needed to fit the SSB-version of PCM is the following:

```
proc nlin data=name-data  maxiter=1000 sigsq=1; like = 0;
PARMS b1_2-b1_k=0 b2_1-b2_k=0 theta_1-theta_n=0;
theta=theta_1*col1+theta_2*col2+...+theta_n*coln; b1_1=0;
beta1=b1_1*COL(n+1)+b1_2*COL(n+2)+...+b1_k*COL(n+k);
beta2=b2_1*COL(n+1)+b2_2*COL(n+2)+...+b2_k*COL(n+k);
exp1=exp(theta-beta1); exp2=exp(2*theta-beta1-beta2);
denom=1+exp1+exp2; p1=1/denom; p2=exp1/denom; p3=exp2/denom;
model like = sqrt(- 2 * ( n1*log(p2)+n2*log(p3)+(nt-n1-n2)*log(p1) ) );
run;
```

Let us remark that, when we have a large number of examinees, it is necessary to implement a SAS-procedure to get the design matrix. For interesting readers, the procedure is available under a request to the first author.

B A class of partitioned matrices

B.1 Introduction

The main purpose of this appendix is to provide an explicit formula for the inverse of a class of partitioned matrices, whose form is motivated by the Fisher information matrices $\mathcal{I}_{\text{JMLE}}$ and $\mathcal{I}_{\text{SSBE}}$ of the PCM model and its SSB-formulation; for examples, see pages 13 and 14. The content of this Appendix is a detailed explanation of what was sketched by del Pino et al. (2008).

Let us introduce some notation to deal with these patterned matrices. Letting $S_k = \{1, \dots, k\}$, an entry $C_{rr'}$ of a squared matrix of order k can be identified with the value of a function defined on $S_k \times S_k$. An integer array $\boldsymbol{n} = (n_1, \dots, n_{T+1})$, with $\sum_{i=1}^{T+1} n_i = k$, induces a partition of S_k:

$$\left(\{1, \dots, n_1\}, \{n_1 + 1, \dots, n_1 + n_2\}, \dots, \left\{ \sum_{i=1}^{T} n_i + 1, \dots, \sum_{i=1}^{T+1} n_i \right\} \right).$$

The increasing order of the integers which appear in each subset of the partition should be respected. In turn this partition induces a block pattern for C, as illustrated by the matrices $\mathcal{I}_{\text{JMLE}}$ and $\mathcal{I}_{\text{JMLE}}^{-1}$ at page 14. Finally, $r \in S_k$ is in one to one correspondence with the pair (t, s), where t labels one subset in the partition and s identifies its s-th element. In this way, we may write $C_{rr'} = c(t, s; t', s')$ for some function c with four integer arguments. Take for instance matrix $C = \mathcal{I}_{\text{JMLE}}^{-1}$ at page 14. There $\boldsymbol{n} = (2, 1, 4, 2, 1, 9)$ and $T = 5$. The element $C_{49} = 0.870$ corresponds to the third subset of the partition ($t = 2$) and its first element ($s = 1$), and to the fourth subset of the partition ($t' = 4$) and its first element ($s' = 1$). Therefore, we write $C_{49} = c(3, 2; 4, 2)$.

B.2 The class $\mathcal{C}(\boldsymbol{n})$

The blocks of the patterned matrix as those of pages 13 and 14 satisfy a large number of equality constraints. The behavior is different between $t \leq T$ and $t = T + 1$, and so we rewrite the local index s as j, and similarly s' as j'. We also write $n = \sum_{i=1}^{T} n_i$ and $m = n_{T+1}$. With this convention, the constraints essentially convey the idea that the local indices (s, s') do not affect the entry values of the matrix. There is however one exception, namely that for $t = t'$, it is relevant whether these local indices are equal or not.

An alternative way of expressing this idea is that the values of the function c may be written as values of functions with fewer arguments, and this is what is done in the following definition.

Definition B.1 *Let $t \neq t' \leq T$. A square matrix C of order $m + n$ is said to belong to the class $\mathcal{C}(\boldsymbol{n})$ (or just $C \in \mathcal{C}(\boldsymbol{n})$) if there exist functions d, e, u, v satisfying $c(t, s; t, s) = d(t)$ with $1 \leq s \leq n_t$; $c(t, s; t, s') = e(t, t)$ for $1 \leq s \neq s' \leq n_t$; $c(t, s; t', s') = e(t, t')$ with $1 \leq s \leq n_t$ and $1 \leq s' \leq n_{t'}$; $c(t, s; T+1, j') = u(t, j')$; and $c(T+1, j; t', s') = v(t, j)$. The values $c(T+1, j; T+1, j')$ are arbitrary.*

$\mathcal{C}_S(\boldsymbol{n})$ is the subclass formed by all symmetric matrices in $\mathcal{C}(\boldsymbol{n})$, in which case $v = u$ and $e(t, t') = e(t', t)$. $\mathcal{C}_{S0}(\boldsymbol{n})$ is the subclass of $\mathcal{C}_S(\boldsymbol{n})$ determined by the constraint $e = 0$.

Theorem B.1 *If a nonsingular C belongs to the class $\mathcal{C}(\boldsymbol{n})$ then also its inverse does. The same holds for $\mathcal{C}_S(\boldsymbol{n})$.*

B.3 The bar-operation

Let us introduce the *bar-operation* which transforms a $(n+m)\times(n+m)$ matrix $C \in \mathcal{C}_{S0}(\boldsymbol{n})$ into a square matrix $\overline{C}$ of $(T+m)\times(T+m)$ according to

$$\begin{array}{rcll} \overline{C}_{t,t} &=& n_t d_t, & t \in S_T \\ \overline{C}_{t,t'} &=& 0, & t \neq t' \in S_T \\ \overline{C}_{t,T+j'} &=& n_t\, u(t,j'), & t \in S_T,\ j' \in S_m \\ \overline{C}_{T+j,T+j'} &=& c(T+1,j;T+1,j'), & j,j' \in S_m. \end{array} \tag{B.1}$$

Clearly $\overline{C}$ determines C uniquely. If C is written in the following partitioned form

$$C = \left[\begin{array}{cc} D & A \\ A' & H \end{array}\right], \tag{B.2}$$

with $D_{n\times n}$, $H_{m\times m}$ and $A_{n\times m}$, then $\overline{C}$ is also written in partitioned form by attaching the bar sign to the submatrices, that is,

$$\overline{C}_{(T+m)\times(T+m)} = \left[\begin{array}{cc} \overline{D} & \overline{A} \\ \overline{A}' & \overline{H} \end{array}\right]. \tag{B.3}$$

By definition, $\overline{H} = H$. For instance, let $C = \mathcal{I}_{\text{JMLE}}$ as given at page 14, then $\overline{C}$ is given by

$$\overline{C} = \left(\begin{array}{rrrrr|rrrrrrrrr}
2.248 & 0 & 0 & 0 & 0 & -0.484 & -0.392 & -0.360 & -0.256 & -0.044 & -0.146 & -0.018 & -0.028 & -0.006 \\
0 & 1.444 & 0 & 0 & 0 & -0.317 & -0.123 & -0.256 & -0.206 & -0.053 & -0.190 & -0.020 & -0.045 & -0.013 \\
0 & 0 & 6.440 & 0 & 0 & -1.128 & -0.296 & -1.108 & -1.028 & -0.392 & -1.176 & -0.148 & -0.400 & -0.132 \\
0 & 0 & 0 & 2.712 & 0 & -0.054 & -0.022 & -0.160 & -0.278 & -0.512 & -0.316 & -0.348 & -0.554 & -0.420 \\
0 & 0 & 0 & 0 & 1.150 & -0.006 & -0.005 & -0.027 & -0.064 & -0.229 & -0.081 & -0.238 & -0.225 & -0.265 \\ \hline
-0.484 & -0.317 & -1.128 & -0.054 & -0.006 & 1.501 & 0 & 0 & 0 & 0 & 0.488 & 0 & 0 & 0 \\
-0.392 & -0.123 & -0.296 & -0.022 & -0.005 & 0 & 0.817 & 0 & 0 & 0 & 0 & 0.023 & 0 & 0 \\
-0.360 & -0.256 & -1.108 & -0.160 & -0.027 & 0 & 0 & 1.687 & 0 & 0 & 0 & 0 & 0.221 & 0 \\
-0.256 & -0.206 & -1.028 & -0.278 & -0.064 & 0 & 0 & 0 & 1.697 & 0 & 0 & 0 & 0 & 0.136 \\
-0.044 & -0.053 & -0.392 & -0.512 & -0.229 & 0 & 0 & 0 & 0 & 1.120 & 0 & 0 & 0 & 0 \\
-0.146 & -0.190 & -1.176 & -0.316 & -0.081 & 0.488 & 0 & 0 & 0 & 0 & 1.419 & 0 & 0 & 0 \\
-0.018 & -0.020 & -0.148 & -0.348 & -0.238 & 0 & 0.023 & 0 & 0 & 0 & 0 & 0.749 & 0 & 0 \\
-0.028 & -0.045 & -0.400 & -0.554 & -0.225 & 0 & 0 & 0.221 & 0 & 0 & 0 & 0 & 1.031 & 0 \\
-0.006 & -0.013 & -0.132 & -0.420 & -0.265 & 0 & 0 & 0 & 0.136 & 0 & 0 & 0 & 0 & 0.701
\end{array}\right).$$

The following proposition will be used in the proof of Theorem B.2:

Proposition B.1 *If $C \in \mathcal{C}_S(\boldsymbol{n})$ is positive definite, so is $\overline{C}$.*

B.4 The tilde-operation

Let $C \in \mathcal{C}_S(\boldsymbol{n})$ be a matrix of orden $m+n$ partitioned as (B.2). The tilde-operation is applied to the block A. A matrix $\widetilde{A}$ of order $T \times m$ is obtained, which is defined as follows:

$$\widetilde{A}_{tj'} = u(t,j') \qquad t \in S_T,\ \ j' \in S_m,$$

where $u(t, j') = c(t, s; T+1, j')$ as given at Definition B.1. In order words, $\widetilde{A}_{tj'}$ coincide with the j'-th column of the t block of A. As an example, consider the matrix $C = \mathcal{I}_{\text{JMLE}}^{-1}$ shown at page 14. In this case, $T = 5$ and $m = 9$. Therefore, $\widetilde{A}$ is given by

$$\widetilde{A} = \begin{pmatrix} 0.738 & 0.800 & 0.699 & 0.680 & 0.717 & 0.625 & 0.651 & 0.639 & 0.646 \\ 0.783 & 0.747 & 0.774 & 0.770 & 0.838 & 0.765 & 0.767 & 0.760 & 0.764 \\ 0.780 & 0.740 & 0.818 & 0.838 & 0.955 & 0.890 & 0.878 & 0.888 & 0.886 \\ 0.744 & 0.760 & 0.820 & 0.911 & 1.351 & 1.059 & 1.328 & 1.343 & 1.442 \\ 0.752 & 0.762 & 0.812 & 0.896 & 1.429 & 1.064 & 1.512 & 1.413 & 1.649 \end{pmatrix}.$$

B.5 Main results

The final objective is to obtain a convenient formula for C^{-1} when $C \in \mathcal{C}_{S0}(\boldsymbol{n})$. By Theorem B.1, $C^{-1} \in \mathcal{C}(\boldsymbol{n})$. Let us consider C partitioned as in (B.2) and $\overline{C}$ partitioned as in (B.3). Their corresponding inverses are partitioned as follows:

$$C^{-1} = \begin{bmatrix} X & B \\ B' & W \end{bmatrix}, \quad \overline{C}^{-1} = \begin{bmatrix} E & F \\ F' & S \end{bmatrix},$$

where X is of $n \times n$, B is of $n \times m$, H and S are of $m \times m$, E is of $T \times T$, and F is of $T \times m$.

Theorem B.2 *For the matrices C, C^{-1}, $\overline{C}$ and $\overline{C}^{-1}$, the following relationships are true:*

(i) $W = S$.

(ii) $F = \widetilde{B}$.

(iii) For any nonsingular matrix A, denote by A^{ij} the ij-entry of its inverse. For a given $r \leq n$, let t be determined by:

$$r = \sum_{i=1}^{t-1} n_i + s, \qquad \textit{with } 1 \leq s \leq n_t.$$

Then

$$C^{rr} = \overline{C}^{tt} + \frac{n_t - 1}{n_t d_t}, \qquad \textit{for } r \in S_n. \tag{B.4}$$

(iv) If $C \in \mathcal{C}_{\mathcal{S}_0}(\boldsymbol{n})$ is positive definite then

$$\frac{1}{d_t} \leq C^{rr} \leq n_t \cdot \overline{C}^{tt}, \qquad \textit{for } r \in S_n. \tag{B.5}$$

B.6 Proofs

Proof of Theorem B.1: The first n rows and columns of the matrix C form T natural groups of size $n_t, t \in S_T$. Performing the same permutation P_t for the rows and columns in the t-th group produces a matrix $P_t C P_t'$. But this operation is equivalent to performing a permutation of the elements of $\{\sum_{k=1}^{t-1} n_k + 1, \ldots, \sum_{k=1}^{t} n_k\}$, before evaluating the function $c(t, s; t', s')$. This shows that $C \in \mathcal{C}(\boldsymbol{n})$ if, and only if, $P_t C P_t' = C$ for all such permutations matrices. Since $P_t' = P_t^{-1}$ it follows that

$$P_t C^{-1} P_t' = \left[(P_t')^{-1} C P_t^{-1}\right]^{-1} = (P_t C P_t')^{-1} = C^{-1}.$$

For $\mathcal{C}_S(\boldsymbol{n})$ just use the fact that the inverse of a symmetric matrix is also symmetric.

□

Proof of Proposition B.1: $\boldsymbol{y}_{n+m\times 1}$ is a $\mathcal{C}(\boldsymbol{n})$-vector if their entries are constant within the T sets that partition S_n. This means that there exist a function f satisfying $y_r = f(t)$, for $r \le n$. For $\boldsymbol{y}_{n+m\times 1}$ define $\overline{\boldsymbol{y}}_{T+m\times 1}$ with entries $\overline{y}_i = f(i), i \le T$ and $\overline{y}_{T+j} = y_{n+j}, 1 \le j \le m$. Straightforward computations show that $\boldsymbol{y}' C \boldsymbol{y} = \overline{\boldsymbol{y}} \overline{C} \overline{\boldsymbol{y}}$. Since C is positive definite, $\boldsymbol{y}' C \boldsymbol{y} > 0$, for all $\boldsymbol{y} \neq \boldsymbol{0}$ and it follows that $\overline{\boldsymbol{y}} \overline{C} \overline{\boldsymbol{y}} > 0$ for all $\overline{\boldsymbol{y}} \neq \boldsymbol{0}$.

□

Proof of Theorem B.2: Denote by I_k the identity matrix of order k. Then the key relationships follow from $C\, C^{-1} = I_{(n+m)\times(n+m)}$ and $\overline{C}\, (\overline{C})^{-1} = I_{(T+m)\times(T+m)}$, where the multiplications are performed using the partitioned form.

• *Proof of (i)*: The standard formula for an inverse partitioned matrix yields $W = (H - A' D^{-1} A)^{-1}$ and $S = (H - \overline{A}'(\overline{D})^{-1}\overline{A})^{-1}$. The entries jj' of $A' D^{-1} A$ and $\overline{A}'\overline{D}^{-1}\overline{A}$ are

$$\sum_r \frac{1}{d_t} \widetilde{a}_{tj} \widetilde{a}_{tj'} = \sum_{t=1}^{T} \frac{1}{d_t} \sum_{s=1}^{n_t} \widetilde{a}_{tj} \widetilde{a}_{tj'} = \sum_{t=1}^{T} \frac{n_t}{d_t} \widetilde{a}_{tj} \widetilde{a}_{tj'}$$

and

$$\sum_{t=1}^{T} \frac{1}{n_t d_t} (n_t \widetilde{a}_{tj})(n_t \widetilde{a}_{tj'}) \quad = \quad \sum_{t=1}^{T} \frac{n_t}{d_t} \widetilde{a}_{tj} \widetilde{a}_{tj'},$$

respectively. The equality of these two expressions implies $A' D^{-1} A = \overline{A}'\overline{D}^{-1}\overline{A}$ and hence $W = S$.

• *Proof of (ii)*: From $CC^{-1} = I_{(n+m)\times(n+m)}$ it follows that $(DB + AW)_{ij} = D_{ii} B_{ij} + \sum_{s=1}^{m} A_{is} W_{sj} = d_t b_{tj} + \sum_{s=1}^{m} \widetilde{a}_{ts} W_{sj} = 0$. Similarly, $\overline{C}(\overline{C})^{-1} = I_{(T+m)\times(T+m)}$ and $W = S$ yield the equivalent equation $n_t d_t F_{tj} + \sum_{s=1}^{m} n_t \widetilde{a}_{ts} W_{sj} = 0$. Therefore, $F_{tj} = b_{tj}$ for all t, j. Therefore, $F = \widetilde{B}$.

• *Proof of (iii)*: By definition, $(C^{-1})^{rr} = X_{rr} \equiv d_t^*$ and $[(\overline{C})^{-1}]^{tt} \equiv E_{tt}$. We must then show that $d_t^* = E_{tt} + \frac{n_t - 1}{D_t}$. Now, $DX + AB' = I_{n\times n}$ implies $(DX)_{rr} + (AB')_{rr} = 1$ and so $d_t d_t^* + u(t,t) = 1$, where $u_t = \sum_s \widetilde{a}_{ts} \widetilde{b}_{ts}$. Similarly, $(\overline{D}E) + \overline{A}\, F' = \overline{D}E + \overline{A}\, \widetilde{B}' = I_{T\times T}$, yields $n_t d_t E_{tt} + n_t u(t,t) = 1, t \in S_T$. Eliminating u_t from these two equations, (B.4) is obtained.

• *Proof of (iv)*: Since $C \in \mathcal{C}(\boldsymbol{n})$ is positive definite, then by Proposition B.1 $\overline{C}$ is also positive definite. For any positive define M, the inequality $\frac{1}{M_{ii}} \leq M^{ii}$ holds. Applying it to $(M = C, i = r \leq n)$ and $(M = \overline{C}, t \leq T)$ yields $\frac{1}{d_t} \leq C^{rr}$ and $\frac{1}{n_t d_t} \leq \overline{C}^{tt}$ respectively. Multiplying the second inequality by $n_t - 1$, adding $\overline{C}^{tt}$ to both sides, and using (B.4) the upper bound in (B.5) follows.

□

Acknowledgements: The first autor was partially supported by a VRAID-grant from the Pontificia Universidad Católica de Chile. The second author gratefully acknowledges a partial financial support from the FONDECYT Project N$^{\text{o}}$ 1181261 from the Chilean Government.

References

Adams, R. A. and S. T. Khoo (1993). *QUEST*. Australian Council for Educational Research.

Agresti, A., B. Caffo, and P. Ohman-Strickland (2004). Examples in which misspecification of a random effects distribution reduces efficiency. *Computational Statistics and Data Analysis 47*, 639–653.

Andersen, E. B. (1980). *Discrete Statistical Models with Social Sciences Applications*. North-Holland.

Angoff, W. H. (1971). Scale, norms, and equivalent scores. In R. L. Thorndike (Ed.), *Educational Measurement*, pp. 508–600. American Council of Education.

Baker, F. B. and S. H. Kim (2004). *Item response Theory. Parameter Estimation Techniques*. Marcel Dekker.

Barndorff-Nielsen, O. (1978). *Information and Exponential Families in Statistical Theory*. New York: Wiley.

Berk, R. H. (1970). Consistency a posteriori. *The Annals of Statistics 41*, 894–906.

Bertoli-Barsotti, L. (2005). On the Existence and Uniqueness of JML Estimates for the Partial Credit Model. *Psychometrika 70*, 517–531.

Besag, J. (1974). Spatial interaction and the statistical analysis of lattice systems (with Discussion). *Journal of the Royal Statistical Society, Series B 36*, 192–236.

Besag, J. (1975). Statistical analysis of non-lattice data. *The Statistician 24*, 179–195.

Bravo, D. and J. Manzi (2004). Antecedentes y características del Proyecto SIES. Technical report, Proyecto FONDEF SIES. Available at *www.sectormatematica.cl/ppt/SIES.ppt*.

Cizek, G. J. (2001). *Setting performance standards: concepts, methods, and perspectives*. Lawrence Erlbaum.

De Boeck, P. and M. Wilson (2004). *Explanatory Item Response Models. A Generalized Linear and Nonlinear Approach*. Springer.

del Pino, G., E. San Martín, J. González, and P. De Boeck (2008). On the Relationships Between Sum Score Based Estimation and Joint Maximum Likelihood Estimation. *Psychometrika 73*, 145–151.

Embretson, S. E. and S. P. Reise (2000). *Item Response Theory for Psychologists*. Lawrence Elbaurm Associates.

Engle, R. E., D. F. Hendry, and J. F. Richard (1983). Exogeneity. *Econometrica 51*, 277–304.

Florens, J.-P., M. Mouchart, and J.-M. Rolin (1990a). *Elements of Bayesian Statistics*. Marcel Dekker.

Florens, J.-P., M. Mouchart, and J.-M. Rolin (1990b). *Elements of Bayesian Statistics*. Marcel Dekker.

Ghosh, M. (1995). Inconsistent maximum likelihood estimators for the Rasch model. *Statistical and Probability Letters 23*, 165–170.

Glaser, R. (1963). Instructional technology and the measurement of learning outcomes. *American Psychologist 18*, 519–521.

Haberman, S. J. (1977). Maximum Likelihood Estimates in Exponential Response Models. *The Annals of Statistics 5*, 815–841.

Heagerty, P. J. and B. F. Kurland (2001). Misspecified maximum likelihood estimates and generalized linear mixed models. *Biometrika 88*, 973–985.

Henze, N. (1986). A Probabilistic Representation of the 'Skew-normal' Distribution. *Scandinavian Journal of Statistics 13*, 271–275.

Janssen, R., F. Tuerlinckx, and P. De Boeck (2000). A Hierarchical IRT Model for Criterior-Reference Measurement. *Journal of Educational and Behavioral Statistics 25*, 285–306.

Kiefer, J. and J. Wolfowitz (1956). Consistency of the Maximum Likelihood Estimators in the Presence of Infinitely Many Incidental Parameters. *The Annals of Mathematical Statistics 27*, 887–906.

Lancaster, T. (2000). The incidental parameter problem since 1948. *Journal of Econometrics 95*, 391–413.

Laplace, P. S. (1820). *Théorie Analytique des Probabilités*. Paris: Mme. Ve. Courcier, Imprimeur-Libraire pour les Mathématiques.

Lazarsfeld, P. F. (1950). The Logical and Mathematical Foundation of Llatent Structural Analysis. In S. S. Stout (Ed.), *Measurement and Prediction*, pp. 362–412. Science Editions.

Linacre, J. M. and B. D. Wright (2000). *WINSTEPS: Multiple-choice, Rating Scale and Partial Credit Rasch Analysis*. MESA Press.

Livinston, S. and M. Zieky (1989). A comparative study of standard-setting methods. *Applied Measurement in Education 2*, 121–141.

Master, G. N. (1982). A rasch model for partial credit scoring. *Psychometrika 47*, 149–174.

Master, G. N. and B. D. Wright (1997). The Partial Credit Model. In W. J. van der Linden and R. K. Hambleton (Eds.), *Handbook of Modern Item Response Theory*, pp. 101–121. Springer.

McCullagh, P. and J. A. Nelder (1989). *Generalized linear models*. Chapmann & Hall.

Meskauskas, J. A. (1976). Evaluation models for criterion-referenced testing: Views regarding mastery in standard setting. *Review of Educational Research 45*, 133–158.

Mouchart, M. and E. San Martin (2003). Specification and identification issues in models involving a latent hierarchical structure. *Journal of Statistical Planning and Inference 111*, 143–163.

Neyman, J. and E. Scott (1948). Consistent Estimates Based on Partially Consistent Observations. *Econometrica 16*, 1–32.

Reichenbach, H. (1956). *The direction of time*. University CaliforniaPress, Berkeley.

San Martín, E., J. González, and F. Tuerlinckx (2009). Identified Paremeters, Parameters of Interest and Their Relationships. *Measurement: Interdisciplinary Research and Perspective*.

San Martín, E. and J. M. Rolin (2007). Identifiability and Estimability of Parametric Rasch-Type Models. *Submitted*.

Verbeke, G. and E. Lesaffre (1996). A linear mixed-effects model with heterogeneity in the random effects population. *Journal of the American Statistical Association 91*, 217–221.

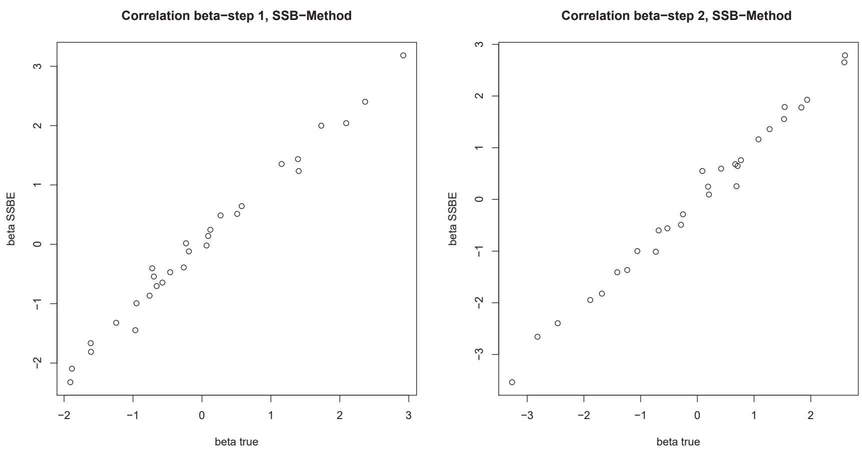

Figure 1: Mixture of Normal Distributions with different weights, β-estimates

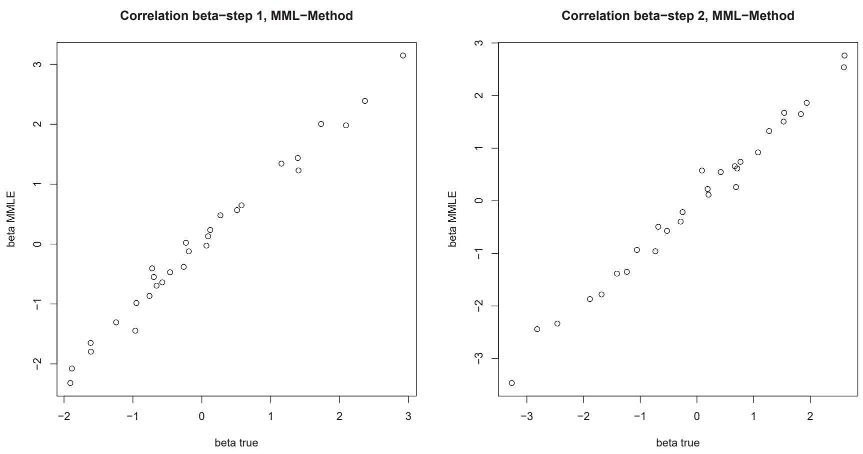

Figure 2: Mixture of Normal Distributions with different weights, β-estimates

Table 1: Data matrix used for fitting the SSB-model $n = 10$, $k = 5$ and $h = 2$

t	i	n_t	n_{ti1}	n_{ti2}
2	1	2	1	0
2	2	2	1	0
2	3	2	2	0
2	4	2	0	0
2	5	2	0	0
3	1	1	1	0
3	2	1	0	0
3	3	1	0	0
3	4	1	0	1
3	5	1	0	0
4	1	4	3	0
4	2	4	2	1
4	3	4	4	0
4	4	4	3	0
4	5	4	2	0
7	1	2	1	1
7	2	2	0	2
7	3	2	2	0
7	4	2	1	1
7	5	2	2	0
8	1	1	0	1
8	2	1	0	1
8	3	1	0	1
8	4	1	0	0
8	5	1	0	1

Table 2: Computation of $(s.e.(\widehat{\theta}_v))^2$ from $(s.e.(\widehat{\gamma}_t))^2$ with $v \in I_t$ using (4.4)

t	n_t	$[\mathcal{I}_{\text{SSBE}}]_{tt}$	$(s.e.(\widehat{\gamma}_t))^2$	$(n_t - 1)/\,[\mathcal{I}_{\text{SSBE}}]_{tt}$	$(s.e.(\widehat{\theta}_v))^2$
2	2	2.247	1.002	0.445	1.447
3	1	1.444	1.348	0	1.348
4	4	6.440	0.914	0.466	1.380
7	2	2.713	1.577	0.369	1.945
8	1	1.150	2.277	0	2.277

Table 3: JMLE and SSBE for the difficulty parameters, along with the corresponding estimated standard errors

	JMLE				SSBE			
item i	$\widehat{\beta}_{i1}$	$s.e(\widehat{\beta}_{i1})$	$\widehat{\beta}_{i2}$	$s.e(\widehat{\beta}_{i2})$	$\widehat{\beta}_{i1}$	$s.e(\widehat{\beta}_{i1})$	$\widehat{\beta}_{i2}$	$s.e.(\widehat{\beta}_{i2})$
1	0	0	$-0,5619$	$0,8221$	0	0	$-0,5619$	$0,8221$
2	$-1,0689$	$0,4921$	$-0,1059$	$0,6109$	$-1,0689$	$0,4921$	$-0,1059$	$0,6109$
3	$0,0278$	$0,5929$	$-2,3078$	$0,6120$	$0,0278$	$0,5929$	$-2,3078$	$0,6120$
4	$-0,0178$	$0,5942$	$-2,3738$	$0,6092$	$-0,0178$	$0,5942$	$-2,3738$	$0,6092$
5	$-1,0076$	$0,5006$	$-0,6528$	$0,5691$	$-1,0076$	$0,5006$	$-0,6528$	$0,5691$
6	$-1,2177$	$0,5026$	$-0,8168$	$0,5471$	$-1,2177$	$0,5026$	$-0,8168$	$0,5471$
7	$-1,5700$	$0,5519$	$-1,9611$	$0,5162$	$-1,5700$	$0,5519$	$-1,9611$	$0,5162$
8	$-2,4255$	$0,5070$	$0,8630$	$0,6560$	$-2,4255$	$0,5070$	$0,8630$	$0,6560$
9	$1,2121$	$0,7117$	$-1,9922$	$0,7932$	$1,2121$	$0,7117$	$-1,9922$	$0,7932$
10	$0,5681$	$0,6129$	$-1,7677$	$0,6803$	$0,5681$	$0,6129$	$-1,7677$	$0,6803$
11	$1,2468$	$0,7113$	$-1,8245$	$0,8084$	$1,2468$	$0,7113$	$-1,8245$	$0,8084$
12	$-0,7356$	$0,5960$	$-2,7000$	$0,5724$	$-0,7356$	$0,5960$	$-2,7000$	$0,5724$
13	$-0,5593$	$0,5252$	$-1,3253$	$0,5693$	$-0,5593$	$0,5252$	$-1,3253$	$0,5693$
14	$0,8149$	$0,8344$	$-3,9363$	$0,8275$	$0,8149$	$0,8344$	$-3,9363$	$0,8275$
15	$-0,7016$	$0,5298$	$-1,5768$	$0,5538$	$-0,7016$	$0,5298$	$-1,5768$	$0,5538$
16	$-0,6423$	$0,5180$	$-1,1452$	$0,5695$	$-0,6423$	$0,5180$	$-1,1452$	$0,5695$
17	$-0,6083$	$0,5689$	$-2,3268$	$0,5645$	$-0,6083$	$0,5689$	$-2,3268$	$0,5645$
18	$1,1025$	$0,6492$	$0,3241$	$1,1860$	$1,1025$	$0,6492$	$0,3241$	$1,1860$
19	$-1,0125$	$0,5551$	$-2,1627$	$0,5399$	$-1,0125$	$0,5551$	$-2,1627$	$0,5399$
20	$-0,8679$	$0,4963$	$-0,2147$	$0,6168$	$-0,8679$	$0,4963$	$-0,2147$	$0,6168$
21	$-1,4417$	$0,4851$	$0,5449$	$0,6668$	$-1,4417$	$0,4851$	$0,5449$	$0,6668$
22	$1,5424$	$0,8211$	$-2,7872$	$0,8670$	$1,5424$	$0,8211$	$-2,7872$	$0,8670$
23	$-0,6993$	$0,5039$	$-0,5048$	$0,6045$	$-0,6993$	$0,5039$	$-0,5048$	$0,6045$
24	$-0,4595$	$0,5469$	$-1,8437$	$0,5681$	$-0,4595$	$0,5469$	$-1,8437$	$0,5681$
25	$1,8165$	$0,8191$	$-1,0654$	$1,0754$	$1,8165$	$0,8191$	$-1,0654$	$1,0754$
26	$1,1771$	$0,7120$	$-2,1387$	$0,7815$	$1,1771$	$0,7120$	$-2,1387$	$0,7815$
27	$0,5681$	$0,6129$	$-1,7677$	$0,6803$	$0,5681$	$0,6129$	$-1,7677$	$0,6803$
28	$0,9957$	$0,7144$	$-2,6822$	$0,7488$	$0,9957$	$0,7144$	$-2,6822$	$0,7488$
29	$1,3155$	$0,7108$	$-1,3904$	$0,8591$	$1,3155$	$0,7108$	$-1,3904$	$0,8591$
30	$0,3147$	$0,5634$	$-0,8028$	$0,7043$	$0,3147$	$0,5634$	$-0,8028$	$0,7043$

Table 4: JMLE and SSBE for the ability parameters, and the corresponding standard errors

t	n_t	$\widehat{\gamma}_t$	$s.e.(\widehat{\gamma}_t)$	$\widehat{\theta}_i$	$s.e.(\widehat{\theta}_v)$	$s.e.(\widehat{\theta}_v)/s.e.(\widehat{\gamma}_t)$	$\sqrt{n_t}$
2	1	$-3,4443$	$0,7709$	$-3,4443$	$0,7709$	$1,0000$	$1,0000$
6	1	$-2,4757$	$0,5373$	$-2,4757$	$0,5373$	$1,0000$	$1,0000$
9	1	$-2,1145$	$0,4934$	$-2,1145$	$0,4934$	$1,0000$	$1,0000$
10	2	$-2,0178$	$0,4342$	$-2,0178$	$0,4846$	$1,1161$	$1,4142$
11	2	$-1,9287$	$0,4303$	$-1,9287$	$0,4775$	$1,1097$	$1,4142$
12	1	$-1,8457$	$0,4717$	$-1,8457$	$0,4717$	$1,0000$	$1,0000$
13	1	$-1,7678$	$0,4668$	$-1,7678$	$0,4668$	$1,0000$	$1,0000$
14	3	$-1,6940$	$0,4076$	$-1,6940$	$0,4627$	$1,1352$	$1,7321$
15	5	$-1,6238$	$0,3948$	$-1,6238$	$0,4592$	$1,1631$	$2,2361$
16	3	$-1,5565$	$0,4051$	$-1,5565$	$0,4562$	$1,1261$	$1,7321$
17	4	$-1,4917$	$0,3976$	$-1,4917$	$0,4536$	$1,1408$	$2,0000$
18	2	$-1,4291$	$0,4159$	$-1,4291$	$0,4514$	$1,0854$	$1,4142$
19	3	$-1,3684$	$0,4026$	$-1,3684$	$0,4495$	$1,1165$	$1,7321$
20	4	$-1,3092$	$0,3960$	$-1,3092$	$0,4479$	$1,1311$	$2,0000$
21	5	$-1,2514$	$0,3920$	$-1,2514$	$0,4465$	$1,1390$	$2,2361$
22	1	$-1,1948$	$0,4453$	$-1,1948$	$0,4453$	$1,0000$	$1,0000$
23	3	$-1,1392$	$0,4008$	$-1,1392$	$0,4443$	$1,1085$	$1,7321$
28	1	$-0,8709$	$0,4418$	$-0,8709$	$0,4418$	$1,0000$	$1,0000$
31	1	$-0,7133$	$0,4420$	$-0,7133$	$0,4420$	$1,0000$	$1,0000$
32	1	$-0,6607$	$0,4424$	$-0,6607$	$0,4424$	$1,0000$	$1,0000$
33	2	$-0,6077$	$0,4119$	$-0,6077$	$0,4429$	$1,0753$	$1,4142$
37	1	$-0,3911$	$0,4465$	$-0,3911$	$0,4465$	$1,0000$	$1,0000$
38	1	$-0,3349$	$0,4479$	$-0,3349$	$0,4479$	$1,0000$	$1,0000$
39	1	$-0,2777$	$0,4495$	$-0,2777$	$0,4495$	$1,0000$	$1,0000$

Table 5: True item-parameters for the simulation study

Item	step 1	step 2	Item	step 1	step 2
1	0	-0,8846	16	-0,9026	-1,2425
2	-0,5737	-0,6807	17	0,5005	-2,6542
3	-0,8371	-1,739	18	0,5912	-1,115
4	0,3182	-2,2671	19	-0,6525	-1,4034
5	-0,1427	-1,4991	20	-0,842	-0,6935
6	-1,7522	-0,2252	21	-1,7916	-0,0251
7	-1,176	-2,1133	22	0,5752	-2,4872
8	-2,7895	0,4789	23	-0,1844	-0,959
9	0,1304	-1,2041	24	-0,4154	-1,6914
10	0,1971	-0,7849	25	1,6767	-0,6138
11	1,5895	-2,0216	26	2,1879	-3,4323
12	-0,2968	-2,7038	27	1,4663	-2,7424
13	-0,9942	-0,7677	28	0,5005	-2,6542
14	0,7367	-4,0345	29	0,6224	-0,8203
15	-1,1721	-1,4317	30	-0,0345	-1,0737

Table 6: Mixture of Normal Distributions with different weights

	Item 18			
	MML		SSBE	
True	Fail	Pass	Fail	Pass
Fail	88.53%	11.47%	98.64%	1.51%
Pass	0%	100%	7.75%	95.35%
	Item 3			
True	Fail	Pass	Fail	Pass
Fail	72.12%	27.88%	97.48%	3.14%
Pass	0%	100%	5.90%	95.28%

Table 7: Mixture of Normal Distributions with equal weights

	Item 18			
	MML		SSBE	
True	Fail	Pass	Fail	Pass
Fail	87.84%	12.16%	96.42%	3.97%
Pass	0%	100%	7.63%	95.35%
	Item 3			
True	Fail	Pass	Fail	Pass
Fail	86.92%	13.08%	97.48%	3.67%
Pass	0%	100%	4.96%	97.52%

Table 8: Normal Distribution

	Item 18			
	MML		SSBE	
True	Fail	Pass	Fail	Pass
Fail	75.49%	24.51%	87.34%	12.92%
Pass	1.94%	99.05%	7.25%	93.62%
	Item 3			
True	Fail	Pass	Fail	Pass
Fail	75.49%	24.51%	87.34%	12.92%
Pass	1.94%	99.05%	7.25%	93.62%

Table 9:

Item	$\beta True$			*MML*				*SSBE*			
	step1	*step2*		$\widehat{\beta}_{i1}$	*S.D.*	$\widehat{\beta}_{i1}$	*S.D.*	$\widehat{\beta}_{i1}$	*S.D.*	$\widehat{\beta}_{i2}$	*S.D.*
1	0	0,8501	*MexIG*	0,0000	0,2562	0,8321	0,2267	0,0000	0,0000	0,7984	0,4060
			MexDI	0,0000	0,2238	0,8574	0,2169	0,0000	0,0000	0,7623	0,3702
			SkwePO	0,0000	0,1935	0,8094	0,2090	0,0000	0,0000	0,7556	0,3493
			SkweNE	0,0000	0,3029	0,9014	0,2114	0,0000	0,0000	0,7453	0,4342
2	0,5737	−0,2039	*MexIG*	0,5764	0,2557	−0,2117	0,2092	0,6313	0,3509	−0,2142	0,3169
			MexDI	0,5990	0,2186	−0,1727	0,1998	0,6074	0,2977	−0,1679	0,2868
			SkwePO	0,5596	0,1844	−0,2129	0,1911	0,5696	0,2621	−0,2092	0,2664
			SkweNE	0,6313	0,3333	−0,2342	0,1929	0,5267	0,4385	−0,2063	0,3470
3	0,2634	1,0583	*MexIG*	0,3357	0,3397	1,0260	0,2454	0,3312	0,4210	1,0544	0,3399
			MexDI	0,2825	0,2732	1,0727	0,2214	0,3009	0,3397	1,1131	0,3011
			SkwePO	0,2492	0,2299	1,1233	0,1959	0,2857	0,2967	1,1407	0,2699
			SkweNE	0,3497	0,6043	1,0724	0,2723	0,3614	0,6678	1,1008	0,3961
4	−1,1553	0,5281	*MexIG*	−1,2879	0,3588	0,5663	0,3185	−1,3315	0,4275∗	0,5640	0,3955
			MexDI	−1,2405	0,3051	0,5409	0,2842	−1,2510	0,3657	0,5331	0,3501
			SkwePO	−1,1586	0,2599	0,4829	0,2502	−1,1839	0,3199	0,4737	0,3115
			SkweNE	−1,1833	0,4873	0,4700	0,3383	−1,1839	0,5643	0,4699	0,4441
5	0,4609	−0,768	*MexIG*	0,5759	0,2974	−0,7614	0,2485	0,5519	0,3751	−0,8212	0,3432
			MexDI	0,5319	0,2480	−0,8369	0,2266	0,5238	0,3198	−0,8418	0,3056
			SkwePO	0,4221	0,2132	−0,7677	0,2093	0,4148	0,2832	−0,7723	0,2798
			SkweNE	0,4514	0,4090	−0,7184	0,2507	0,4476	0,4983	−0,7274	0,3818
6	1,6095	−1,2739	*MexIG*	1,5983	0,2988	−1,2758	0,1870	1,5794	0,3771	−1,2751	0,3035
			MexDI	1,6253	0,2377	−1,2579	0,1806	1,6310	0,3121	−1,2759	0,2737
			SkwePO	1,6436	0,1969	−1,2829	0,1733	1,6650	0,2718	−1,2791	0,2540
			SkweNE	1,5510	0,4566	−1,2824	0,1706	1,5423	0,5383	−1,3093	0,3353
7	−0,5762	1,8881	*MexIG*	−0,4493	0,4242	1,8530	0,2647	−0,4481	0,4797	1,8858	0,3538
			MexDI	−0,5223	0,3155	1,9233	0,2323	−0,5025	0,3749	1,9687	0,3090
			SkwePO	−0,5837	0,2636	1,9590	0,2043	−0,5436	0,3248	1,9805	0,2761
			SkweNE	−0,4936	0,7749∗	1,8758	0,3100	−0,4825	0,8244∗	1,9177	0,4228
8	1,6135	−2,5922	*MexIG*	1,5125	0,4080	−2,5680	0,1789	1,4617	0,4683	−2,5807	0,3018
			MexDI	1,5932	0,2974	−2,5909	0,1797	1,5845	0,3603	−2,6494	0,2736
			SkwePO	1,6217	0,2426	−2,7085	0,1863	1,6021	0,3080	−2,7286	0,2631
			SkweNE	1,8104	0,6263∗	−2,5467	0,1560	1,5291	0,6894∗	−2,6173	0,3290
9	−2,9199	1,683	*MexIG*	−2,9327	0,2758	1,7190	0,2425	−2,8401	0,3605	1,7105	0,3402
			MexDI	−2,9247	0,2376	1,6696	0,2260	−2,9454	0,3119	1,6818	0,3056
			SkwePO	−2,8964	0,2054	1,7302	0,2146	−2,9501	0,2771	1,7192	0,2837
			SkweNE	−3,2457	0,3339	1,7299	0,2366	−2,9716	0,4389	1,7753	0,3729
10	−0,0667	−0,4192	*MexIG*	−0,0494	0,2507	−0,4649	0,2265	−0,0352	0,3417	−0,5098	0,3295
			MexDI	−0,1323	0,2254	−0,3519	0,2227	−0,1467	0,3030	−0,3666	0,3036
			SkwePO	−0,1049	0,1973	−0,3623	0,2208	−0,1129	0,2711	−0,3668	0,2885
			SkweNE	−0,0646	0,2795	−0,4908	0,2073	−0,0732	0,3994	−0,5100	0,3555
11	−1,3924	1,2367	*MexIG*	−1,4604	0,3697	1,2582	0,3592	−1,4244	0,4385	1,2720	0,4305
			MexDI	−1,5627	0,3616	1,3338	0,3626	−1,5695	0,4145	1,3390	0,4172
			SkwePO	−1,3614	0,3058	1,1473	0,3228	−1,3698	0,3578	1,1443	0,3724
			SkweNE	−1,3681	0,3844	1,2492	0,3477	−1,3652	0,4787	1,2611	0,4518
12	1,8863	0,6822	*MexIG*	1,7928	0,4410	0,7792	0,3580	1,7887	0,4958	0,7662	0,4253
			MexDI	2,0333	0,3413	0,5334	0,2946	2,0677	0,3966	0,5947	0,3581
			SkwePO	1,8616	0,2887	0,7478	0,2544	1,9360	0,3447	0,7825	0,3149
			SkweNE	1,8814	0,7454∗	0,7435	0,4467∗	1,9123	0,7957∗	0,8029	0,5324∗
13	0,6974	−1,9361	*MexIG*	0,8804	0,2786	−2,0574	0,2033	0,7980	0,3617	−2,0063	0,3135
			MexDI	0,7740	0,2275	−1,9935	0,1920	0,7604	0,3042	−2,0406	0,2811
			SkwePO	0,7505	0,1915	−1,9610	0,1791	0,7151	0,2675	−1,9801	0,2579
			SkweNE	0,7701	0,4054	−2,0321	0,1940	0,7453	0,4955	−2,0864	0,3475
14	−1,7309	3,2668	*MexIG*	−1,9609	0,7755∗	3,5564	0,7162∗	−1,9443	0,8042∗	3,4999	0,7418∗
			MexDI	−1,7686	0,5534	3,3200	0,5202	−1,7529	0,5892	3,3775	0,5586
			SkwePO	−1,9372	0,5108	3,4125	0,4888	−1,8901	0,5447	3,4382	0,5229
			SkweNE	−1,6628	1,0223∗	3,1657	0,7564∗	−1,6503	1,0615∗	3,1996	0,8090∗
15	1,9088	−2,6028	*MexIG*	2,0927	0,3302	−2,9195	0,2244	2,0300	0,4023	−2,8362	0,3284
			MexDI	1,9075	0,2662	−2,5744	0,2062	1,9086	0,3341	−2,6040	0,2902
			SkwePO	2,1250	0,2196	−2,7724	0,1798	2,1018	0,2890	−2,7873	0,2584
			SkweNE	1,8065	0,6024	−2,4456	0,2445	1,8015	0,6661	−2,4619	0,3776
16	−0,2695	−0,1892	*MexIG*	−0,2901	0,3031	−0,1395	0,2228	−0,2503	0,3840	−0,1191	0,3287
			MexDI	−0,2435	0,2471	−0,2179	0,2038	−0,2439	0,3190	−0,2312	0,2888
			SkwePO	−0,3528	0,2047	−0,1727	0,1842	−0,3706	0,2772	−0,1797	0,2615
			SkweNE	−0,3255	0,4755	−0,1519	0,2312	−0,3271	0,5542	−0,1562	0,3692
17	−1,4031	1,4117	*MexIG*	−1,3497	0,4042	1,4182	0,3583	−1,2920	0,4629	1,3431	0,4268
			MexDI	−1,5435	0,3520	1,5108	0,3303	−1,5476	0,4057	1,5270	0,3884
			SkwePO	−1,4386	0,3019	1,5299	0,2896	−1,4409	0,3550	1,5322	0,3439
			SkweNE	−1,3820	0,5603	1,3560	0,4125	−1,2349	0,6054	1,2580	0,4875
18	−0,0907	−1,5392	*MexIG*	−0,1619	0,2732	−1,5008	0,2544	−0,2022	0,3680	−1,4482	0,3579
			MexDI	−0,0244	0,2449	−1,6289	0,2450	−0,0451	0,3178	−1,6845	0,3204
			SkwePO	−0,0471	0,2170	−1,6592	0,2406	−0,0855	0,2857	−1,6792	0,3039
			SkweNE	−0,0637	0,2946	−1,5566	0,2319	−0,2242	0,4101	−1,4837	0,3703
19	1,2437	0,2884	*MexIG*	1,2220	0,3023	0,2593	0,2349	1,2589	0,3813	0,2343	0,3332
			MexDI	1,3088	0,2495	0,2653	0,2128	1,3284	0,3209	0,3106	0,2952
			SkwePO	1,2474	0,2110	0,3510	0,1927	1,2862	0,2818	0,3688	0,2675
			SkweNE	1,3317	0,4771	0,2890	0,2395	1,3451	0,5555	0,3153	0,3745
20	0,1895	−0,7099	*MexIG*	0,2608	0,2677	−0,7363	0,2044	0,2372	0,3537	−0,7488	0,3140
			MexDI	0,1811	0,2220	−0,7328	0,1945	0,1770	0,3002	−0,7551	0,2830
			SkwePO	0,1955	0,1874	−0,7349	0,1841	0,1842	0,2643	−0,7414	0,2615
			SkweNE	0,2010	0,3881	−0,6795	0,1934	0,1948	0,4814	−0,6958	0,3472

Table 10:

Item	$\beta True$			*MML*				*SSBE*			
	step1	*step2*		$\widehat{\beta}_{i1}$	*S.D.*	$\widehat{\beta}_{i1}$	*S.D.*	$\widehat{\beta}_{i1}$	*S.D.*	$\widehat{\beta}_{i2}$	*S.D.*
21	0,9496	−0,6684	*MexIG*	1,0112	0,3003	−0,6536	0,1841	0,9535	0,3805	−0,6270	0,3049
			MexDI	0,9309	0,2331	−0,6471	0,1811	0,9301	0,3087	−0,6581	0,2742
			SkwePO	0,9799	0,1942	−0,7446	0,1784	0,9951	0,2698	−0,7421	0,2575
			SkweNE	0,9130	0,4061*	−0,6658	0,1665	0,6649	0,4953*	−0,6862	0,3335
22	−2,3668	2,4621	*MexIG*	−2,4751	0,3907	2,5109	0,3521	−2,4213	0,4519	2,4811	0,4230
			MexDI	−2,4125	0,3277	2,4774	0,3095	−2,4006	0,3848	2,5021	0,3710
			SkwePO	−2,3835	0,2800	2,5046	0,2721	−2,3968	0,3364	2,5065	0,3293
			SkweNE	−2,5898	0,5332	2,5931	0,4170	−2,3323	0,6044	2,6389	0,5066
23	0,7596	−1,5282	*MexIG*	0,6556	0,2638	−1,5282	0,2290	0,6658	0,3511	−1,5289	0,3308
			MexDI	0,7923	0,2264	−1,5341	0,2133	0,7764	0,3035	−1,5540	0,2964
			SkwePO	0,8015	0,1931	−1,5350	0,2010	0,7878	0,2682	−1,5436	0,2736
			SkweNE	0,9451	0,3257	−1,7311	0,2087	0,9340	0,4328	−1,7580	0,3560
24	0,231	0,7324	*MexIG*	0,3567	0,3248	0,7570	0,2585	0,3528	0,3969	0,7581	0,3495
			MexDI	0,2935	0,2595	0,6890	0,2255	0,3086	0,3287	0,7094	0,3045
			SkwePO	0,1647	0,2199	0,7350	0,2039	0,1861	0,2885	0,7458	0,2757
			SkweNE	0,1794	0,4679	0,7979	0,2703	0,1888	0,5476	0,8197	0,3948
25	−2,0921	−1,0776	*MexIG*	−2,0086	0,2729	−1,1504	0,2769	−1,9941	0,3613	−1,2759	0,3680
			MexDI	−2,1360	0,2699	−1,0068	0,2938	−2,2074	0,3387	−1,1021	0,3609
			SkwePO	−2,0877	0,2768	−1,0780	0,3541	−2,1433	0,3333	−1,1069	0,4002
			SkweNE	−2,1544	0,2507	−1,0388	0,2444	−2,2058	0,3808	−1,1462	0,3802
26	−0,5112	2,8185	*MexIG*	−0,6409	0,6507*	2,8987	0,6384*	−0,6270	0,6897*	3,0392	0.6800*
			MexDI	−0,6615	0,5888	2,9323	0,5843	−0,6088	0,6224	3,0326	0,6193
			SkwePO	−0,3035	0,4324	2,6428	0,4344	−0,2740	0,4706	2,6603	0,4723
			SkweNE	−0,4332	0,6859	2,7548	0,6434	−0,3837	0,7426	2,8616	0,7049
27	0,7216	−0,6899	*MexIG*	0,6322	0,4725	−0,6357	0,4549	0,6241	0,5239	−0,6656	0,5112
			MexDI	0,8316	0,3980	−0,7974	0,3908	0,8377	0,4463	−0,8099	0,4414
			SkwePO	0,5640	0,3414	−0,5320	0,3435	0,5648	0,3888	−0,5309	0,3905
			SkweNE	0,7537	0,5043	−0,7231	0,4417	0,7538	0,5790	−0,7249	0,5271
28	0,9658	−0,0882	*MexIG*	1,1173	0,4053	−0,2357	0,3612	1,1114	0,4654	−0,2018	0,4304
			MexDI	0,9973	0,3476	−0,0533	0,3241	1,0067	0,4018	−0,0334	0,3831
			SkwePO	0,9594	0,2892	−0,0512	0,2765	0,9857	0,3442	−0,0390	0,3330
			SkweNE	0,9448	0,6048	0,0406	0,4529	0,9500	0,6683	0,0508	0,5365
29	−0,1219	−1,8339	*MexIG*	−0,1490	0,2568	−1,8137	0,2429	−0,1956	0,3466	−1,8856	0,3425
			MexDI	−0,1060	0,2328	−1,8859	0,2377	−0,1444	0,3089	−1,9398	0,3152
			SkwePO	−0,1421	0,2105	−1,8184	0,2434	−0,1835	0,2807	−1,8404	0,3063
			SkweNE	−0,1204	0,2651	−1,9357	0,2145	−0,1444	0,3896	−1,9869	0,3600
30	0,6569	0,2534	*MexIG*	0,7018	0,2647	0,1929	0,2294	0,6964	0,3540	0,2647	0,3341
			MexDI	0,6151	0,2337	0,2833	0,2217	0,6392	0,3090	0,3140	0,3024
			SkwePO	0,6803	0,1974	0,1853	0,2072	0,6968	0,2712	0,1935	0,2782
			SkweNE	0,6275	0,3243	0,2515	0,2195	0,6391	0,4317	0,2752	0,3624

Table 11:

		MML		SSBE	
t		$\widehat{\theta}_t$	$S.D.$	$\widehat{\theta}_t$	$S.D.$
1	$MexIG$				
	$MexDI$	$-2,6869$	$0,1078$	$-3,9800$	$0,2489$
	$SkwePO$	$-1,7888$	$0,1472$	$-3,9582$	$0,3847$
	$SkweNE$				
2	$MexIG$				
	$MexDI$	$-2,5873$	$0,0838$	$-3,5400$	$0,1285$
	$SkwePO$	$-1,6567$	$0,1723$	$-3,3509$	$0,3491$
	$SkweNE$				
3	$MexIG$				
	$MexDI$	$-2,3279$	$0,1728$	$-2,9955$	$0,2541$
	$SkwePO$	$-1,5563$	$0,1995$	$-2,9806$	$0,3185$
	$SkweNE$				
4	$MexIG$				
	$MexDI$	$-2,1734$	$0,1898$	$-2,8116$	$0,2662$
	$SkwePO$	$-1,4894$	$0,2072$	$-2,8215$	$0,3082$
	$SkweNE$				
5	$MexIG$				
	$MexDI$	$-2,0845$	$0,2208$	$-2,6051$	$0,2914$
	$SkwePO$	$-1,3930$	$0,2370$	$-2,6438$	$0,3430$
	$SkweNE$				
6	$MexIG$	$-2,5546$	$0,0666$	$-2,4751$	$0,0911$
	$MexDI$	$-1,9974$	$0,2247$	$-2,5240$	$0,2885$
	$SkwePO$	$-1,3105$	$0,2393$	$-2,4477$	$0,3266$
	$SkweNE$				
7	$MexIG$	$-2,4524$	$0,1087$	$-2,3666$	$0,1465$
	$MexDI$	$-1,8780$	$0,2410$	$-2,3736$	$0,3008$
	$SkwePO$	$-1,2369$	$0,2412$	$-2,3150$	$0,3039$
	$SkweNE$				
8	$MexIG$	$-2,3573$	$0,1136$	$-2,2854$	$0,1571$
	$MexDI$	$-1,7799$	$0,2278$	$-2,2957$	$0,2918$
	$SkwePO$	$-1,1622$	$0,2466$	$-2,2073$	$0,3100$
	$SkweNE$				
9	$MexIG$	$-2,2818$	$0,1012$	$-2,2252$	$0,1408$
	$MexDI$	$-1,7146$	$0,2277$	$-2,1198$	$0,2670$
	$SkwePO$	$-1,0963$	$0,2246$	$-2,0719$	$0,2686$
	$SkweNE$				
10	$MexIG$	$-2,1216$	$0,0897$	$-1,9852$	$0,1233$
	$MexDI$	$-1,6354$	$0,2320$	$-2,0831$	$0,2743$
	$SkwePO$	$-1,0329$	$0,2192$	$-1,9832$	$0,2599$
	$SkweNE$				
11	$MexIG$	$-2,1288$	$0,1616$	$-1,8992$	$0,2089$
	$MexDI$	$-1,5673$	$0,2328$	$-1,9806$	$0,2669$
	$SkwePO$	$-0,9722$	$0,2235$	$-1,8983$	$0,2634$
	$SkweNE$				
12	$MexIG$	$-2,0712$	$0,1275$	$-1,7655$	$0,1708$
	$MexDI$	$-1,5061$	$0,2601$	$-1,8726$	$0,2882$
	$SkwePO$	$-0,9143$	$0,2375$	$-1,8195$	$0,2673$
	$SkweNE$				
13	$MexIG$	$-2,0085$	$0,1311$	$-1,7898$	$0,1745$
	$MexDI$	$-1,4437$	$0,2347$	$-1,7730$	$0,2808$
	$SkwePO$	$-0,8565$	$0,2307$	$-1,7381$	$0,2600$
	$SkweNE$				
14	$MexIG$	$-1,9248$	$0,1469$	$-1,7008$	$0,1951$
	$MexDI$	$-1,3641$	$0,2418$	$-1,7242$	$0,2680$
	$SkwePO$	$-0,8022$	$0,2322$	$-1,6617$	$0,2549$
	$SkweNE$				
15	$MexIG$	$-1,8728$	$0,1495$	$-1,6073$	$0,2050$
	$MexDI$	$-1,3132$	$0,2426$	$-1,6305$	$0,2717$
	$SkwePO$	$-0,7473$	$0,2194$	$-1,6065$	$0,2347$
	$SkweNE$				
16	$MexIG$	$-1,8192$	$0,1815$	$-1,6022$	$0,2442$
	$MexDI$	$-1,2562$	$0,2395$	$-1,5659$	$0,2751$
	$SkwePO$	$-0,6966$	$0,2167$	$-1,5488$	$0,2374$
	$SkweNE$				
17	$MexIG$	$-1,7511$	$0,1926$	$-1,5201$	$0,2514$
	$MexDI$	$-1,1955$	$0,2337$	$-1,5116$	$0,2602$
	$SkwePO$	$-0,6444$	$0,2114$	$-1,4763$	$0,2256$
	$SkweNE$				
18	$MexIG$	$-1,6864$	$0,2028$	$-1,4502$	$0,2660$
	$MexDI$	$-1,1444$	$0,2381$	$-1,4505$	$0,2630$
	$SkwePO$	$-0,5966$	$0,2211$	$-1,4123$	$0,2417$
	$SkweNE$				
19	$MexIG$	$-1,6547$	$0,2146$	$-1,3863$	$0,2759$
	$MexDI$	$-1,0900$	$0,2376$	$-1,3983$	$0,2697$
	$SkwePO$	$-0,5480$	$0,2167$	$-1,3423$	$0,2262$
	$SkweNE$				
20	$MexIG$	$-1,5941$	$0,1766$	$-1,3501$	$0,2326$
	$MexDI$	$-1,0368$	$0,2284$	$-1,3339$	$0,2558$
	$SkwePO$	$-0,4993$	$0,2172$	$-1,2852$	$0,2317$
	$SkweNE$				

Table 12:

		MML		SSBE	
t		$\widehat{\theta}_t$	$S.D.$	$\widehat{\theta}_t$	$S.D.$
21	$MexIG$	−1, 5494	0, 2020	−1, 2978	0, 2642
	$MexDI$	−0, 9826	0, 2311	−1, 2718	0, 2558
	$SkwePO$	−0, 4528	0, 2201	−1, 2350	0, 2387
	$SkweNE$				
22	$MexIG$	−1, 5084	0, 2147	−1, 2373	0, 2831
	$MexDI$	−0, 9332	0, 2296	−1, 2312	0, 2549
	$SkwePO$	−0, 4046	0, 2120	−1, 1791	0, 2167
	$SkweNE$				
23	$MexIG$	−1, 4298	0, 1908	−1, 1314	0, 2468
	$MexDI$	−0, 8841	0, 2304	−1, 1618	0, 2536
	$SkwePO$	−0, 3596	0, 2084	−1, 1323	0, 2181
	$SkweNE$				
24	$MexIG$	−1, 3785	0, 2122	−1, 0823	0, 2630
	$MexDI$	−0, 8366	0, 2262	−1, 1174	0, 2511
	$SkwePO$	−0, 3144	0, 2143	−1, 0749	0, 2226
	$SkweNE$				
25	$MexIG$	−1, 3368	0, 2206	−1, 0608	0, 2770
	$MexDI$	−0, 7871	0, 2255	−1, 0746	0, 2429
	$SkwePO$	−0, 2702	0, 2155	−1, 0198	0, 2217
	$SkweNE$				
26	$MexIG$	−1, 2972	0, 2295	−1, 0125	0, 2871
	$MexDI$	−0, 7398	0, 2248	−1, 0074	0, 2471
	$SkwePO$	−0, 2256	0, 2148	−0, 9679	0, 2208
	$SkweNE$				
27	$MexIG$	−1, 2474	0, 2243	−0, 9506	0, 2775
	$MexDI$	−0, 6910	0, 2288	−0, 9563	0, 2579
	$SkwePO$	−0, 1812	0, 2143	−0, 9164	0, 2219
	$SkweNE$				
28	$MexIG$	−1, 2087	0, 2145	−0, 9187	0, 2642
	$MexDI$	−0, 6434	0, 2262	−0, 9019	0, 2517
	$SkwePO$	−0, 1397	0, 2096	−0, 8756	0, 2125
	$SkweNE$	−1, 3192	0, 0306	−0, 8589	0, 0528
29	$MexIG$	−1, 1534	0, 2237	−0, 8713	0, 2820
	$MexDI$	−0, 5986	0, 2232	−0, 8509	0, 2499
	$SkwePO$	−0, 0930	0, 2138	−0, 8143	0, 2155
	$SkweNE$	−1, 2520	0, 0302	−0, 9422	0, 0526
30	$MexIG$	−1, 1098	0, 2261	−0, 8190	0, 2725
	$MexDI$	−0, 5477	0, 2260	−0, 7985	0, 2476
	$SkwePO$	−0, 0489	0, 2137	−0, 7634	0, 2176
	$SkweNE$	−1, 2679	0, 0438	−1, 0008	0, 0777
31	$MexIG$	−1, 0532	0, 2214	−0, 7587	0, 2738
	$MexDI$	−0, 5008	0, 2284	−0, 7520	0, 2478
	$SkwePO$	−0, 0049	0, 2138	−0, 7126	0, 2140
	$SkweNE$	−1, 2712	0, 0542	−0, 7723	0, 1005
32	$MexIG$	−1, 0098	0, 2289	−0, 7035	0, 2717
	$MexDI$	−0, 4532	0, 2288	−0, 7010	0, 2547
	$SkwePO$	0, 0392	0, 2141	−0, 6616	0, 2145
	$SkweNE$	−1, 2813	0, 0448	−0, 8098	0, 0801
33	$MexIG$	−0, 9592	0, 2297	−0, 6468	0, 2747
	$MexDI$	−0, 4014	0, 2247	−0, 6491	0, 2416
	$SkwePO$	0, 0839	0, 2124	−0, 6107	0, 2133
	$SkweNE$	−1, 0542	0, 0533	−0, 8436	0, 0885
34	$MexIG$	−0, 9135	0, 2305	−0, 6031	0, 2725
	$MexDI$	−0, 3570	0, 2302	−0, 5980	0, 2463
	$SkwePO$	0, 1280	0, 2152	−0, 5587	0, 2095
	$SkweNE$	−1, 0399	0, 0819	−0, 7770	0, 1429
35	$MexIG$	−0, 8636	0, 2317	−0, 5505	0, 2755
	$MexDI$	−0, 3122	0, 2286	−0, 5484	0, 2484
	$SkwePO$	0, 1729	0, 2160	−0, 5066	0, 2171
	$SkweNE$	−0, 9954	0, 0933	−0, 5210	0, 1572
36	$MexIG$	−0, 8128	0, 2232	−0, 5062	0, 2592
	$MexDI$	−0, 2608	0, 2276	−0, 4939	0, 2442
	$SkwePO$	0, 2181	0, 2170	−0, 4538	0, 2123
	$SkweNE$	−0, 9464	0, 1079	−0, 3131	0, 1744
37	$MexIG$	−0, 7617	0, 2370	−0, 4424	0, 2715
	$MexDI$	−0, 2092	0, 2340	−0, 4395	0, 2441
	$SkwePO$	0, 2637	0, 2181	−0, 4003	0, 2112
	$SkweNE$	−0, 9169	0, 1378	−0, 4599	0, 2279
38	$MexIG$	−0, 7102	0, 2366	−0, 3846	0, 2763
	$MexDI$	−0, 1585	0, 2358	−0, 3838	0, 2500
	$SkwePO$	0, 3099	0, 2194	−0, 3458	0, 2136
	$SkweNE$	−0, 8610	0, 1767	−0, 3492	0, 2871
39	$MexIG$	−0, 6574	0, 2411	−0, 3323	0, 2751
	$MexDI$	−0, 1111	0, 2331	−0, 3325	0, 2501
	$SkwePO$	0, 3567	0, 2210	−0, 2902	0, 2127
	$SkweNE$	−0, 7964	0, 1830	−0, 3038	0, 2897
40	$MexIG$	−0, 6080	0, 2362	−0, 2683	0, 2768
	$MexDI$	−0, 0532	0, 2353	−0, 2793	0, 2535
	$SkwePO$	0, 4042	0, 2227	−0, 2334	0, 2129
	$SkweNE$	−0, 7608	0, 2139	−0, 2165	0, 3247

Table 13:

t		MML $\widehat{\theta}_t$	$S.D.$	$SSBE$ $\widehat{\theta}_t$	$S.D.$
41	$MexIG$	$-0,5470$	$0,2441$	$-0,2164$	$0,2745$
	$MexDI$	$-0,0020$	$0,2405$	$-0,2187$	$0,2506$
	$SkwePO$	$0,4526$	$0,2247$	$-0,1750$	$0,2156$
	$SkweNE$	$-0,7119$	$0,2183$	$-0,1552$	$0,3208$
42	$MexIG$	$-0,4933$	$0,2445$	$-0,1606$	$0,2736$
	$MexDI$	$0,0554$	$0,2410$	$-0,1689$	$0,2558$
	$SkwePO$	$0,5008$	$0,2245$	$-0,1157$	$0,2165$
	$SkweNE$	$-0,6600$	$0,2302$	$-0,0407$	$0,3275$
43	$MexIG$	$-0,4297$	$0,2483$	$-0,1033$	$0,2851$
	$MexDI$	$0,1117$	$0,2493$	$-0,0921$	$0,2607$
	$SkwePO$	$0,5521$	$0,2294$	$-0,0527$	$0,2238$
	$SkweNE$	$-0,6086$	$0,2282$	$0,0142$	$0,3158$
44	$MexIG$	$-0,3753$	$0,2526$	$-0,0204$	$0,2793$
	$MexDI$	$0,1711$	$0,2481$	$-0,0344$	$0,2629$
	$SkwePO$	$0,6036$	$0,2322$	$0,0119$	$0,2286$
	$SkweNE$	$-0,5553$	$0,2384$	$0,0785$	$0,3156$
45	$MexIG$	$-0,3141$	$0,2573$	$0,0409$	$0,2887$
	$MexDI$	$0,2314$	$0,2550$	$0,0458$	$0,2646$
	$SkwePO$	$0,6565$	$0,2352$	$0,0792$	$0,2354$
	$SkweNE$	$-0,4999$	$0,2417$	$0,1462$	$0,3100$
46	$MexIG$	$-0,2454$	$0,2652$	$0,1089$	$0,2924$
	$MexDI$	$0,2892$	$0,2598$	$0,1064$	$0,2692$
	$SkwePO$	$0,7108$	$0,2386$	$0,1497$	$0,2390$
	$SkweNE$	$-0,4426$	$0,2453$	$0,2170$	$0,3096$
47	$MexIG$	$-0,1774$	$0,2681$	$0,1859$	$0,3023$
	$MexDI$	$0,3588$	$0,2624$	$0,1866$	$0,2633$
	$SkwePO$	$0,7649$	$0,2397$	$0,2212$	$0,2538$
	$SkweNE$	$-0,3833$	$0,2492$	$0,2915$	$0,3051$
48	$MexIG$	$-0,1045$	$0,2726$	$0,2666$	$0,2993$
	$MexDI$	$0,4279$	$0,2716$	$0,2580$	$0,2635$
	$SkwePO$	$0,8246$	$0,2465$	$0,3047$	$0,2747$
	$SkweNE$	$-0,3219$	$0,2535$	$0,3704$	$0,3048$
49	$MexIG$	$-0,0284$	$0,2831$	$0,3446$	$0,3152$
	$MexDI$	$0,5021$	$0,2786$	$0,3436$	$0,2709$
	$SkwePO$	$0,8842$	$0,2333$	$0,4001$	$0,2607$
	$SkweNE$	$-0,2579$	$0,2582$	$0,4546$	$0,3046$
50	$MexIG$	$0,0609$	$0,2797$	$0,4432$	$0,3054$
	$MexDI$	$0,5795$	$0,2897$	$0,4368$	$0,2672$
	$SkwePO$	$0,9410$	$0,2174$	$0,4836$	$0,2678$
	$SkweNE$	$-0,1912$	$0,2634$	$0,5453$	$0,3037$
51	$MexIG$	$0,1358$	$0,2993$	$0,5263$	$0,3196$
	$MexDI$	$0,6618$	$0,2991$	$0,5351$	$0,2695$
	$SkwePO$	$1,0077$	$0,2153$	$0,5564$	$0,2674$
	$SkweNE$	$-0,1215$	$0,2690$	$0,6438$	$0,3024$
52	$MexIG$	$0,2386$	$0,3181$	$0,6398$	$0,3257$
	$MexDI$	$0,7502$	$0,3099$	$0,6433$	$0,2566$
	$SkwePO$	$1,0944$	$0,2157$	$0,7270$	$0,2720$
	$SkweNE$	$-0,0485$	$0,2752$	$0,7521$	$0,3035$
53	$MexIG$	$0,3414$	$0,3361$	$0,7622$	$0,3216$
	$MexDI$	$0,8459$	$0,3226$	$0,7644$	$0,2584$
	$SkwePO$	$1,1419$	$0,1608$	$0,8332$	$0,2139$
	$SkweNE$	$0,0282$	$0,2820$	$0,8732$	$0,3048$
54	$MexIG$	$0,4563$	$0,3540$	$0,9001$	$0,2921$
	$MexDI$	$0,9504$	$0,3376$	$0,9026$	$0,2588$
	$SkwePO$	$1,2256$	$0,1134$	$0,8877$	$0,1680$
	$SkweNE$	$0,1091$	$0,2895$	$1,0111$	$0,3069$
55	$MexIG$	$0,5852$	$0,3757$	$1,0616$	$0,2846$
	$MexDI$	$1,0661$	$0,3554$	$1,0647$	$0,2621$
	$SkwePO$	$1,2853$	$0,1011$	$1,0432$	$0,1598$
	$SkweNE$	$0,1945$	$0,2977$	$1,1724$	$0,3098$
56	$MexIG$	$0,7327$	$0,4028$	$1,2433$	$0,2815$
	$MexDI$	$1,1957$	$0,3771$	$1,2622$	$0,2788$
	$SkwePO$	$1,3761$	$0,0748$	$1,2601$	$0,1232$
	$SkweNE$	$0,2851$	$0,3068$	$1,3684$	$0,3160$
57	$MexIG$	$0,9050$	$0,4373$	$1,4961$	$0,2819$
	$MexDI$	$1,3434$	$0,4038$	$1,5181$	$0,3067$
	$SkwePO$	$1,5473$	$0,0452$	$1,3796$	$0,0813$
	$SkweNE$	$0,3812$	$0,3169$	$1,6207$	$0,3245$
58	$MexIG$	$1,1121$	$0,4822$	$1,8556$	$0,2938$
	$MexDI$	$1,5151$	$0,4373$	$1,8868$	$0,4400$
	$SkwePO$				
	$SkweNE$	$0,4836$	$0,3280$	$1,9793$	$0,3488$
59	$MexIG$	$1,3698$	$0,5424$	$2,4865$	$0,3460$
	$MexDI$	$1,7250$	$0,4570$	$2,4848$	$0,6589$
	$SkwePO$				
	$SkweNE$				