Estimation of Parameters and Control of the Model for Two Response Categories.

1. Method I.

Taking

$$(1.1)\quad h_i^{(r)} = \frac{a_i^{(r)}}{n_r}$$

as an estimate of the probability that an individual with a total number of r correct answers gives the right answer to question no. i, we have

$$(1.2)\quad h_i^{(r)} \approx \frac{\theta^{(r)}\varepsilon_i}{1+\theta^{(r)}\varepsilon_i}.$$

It follows that the logistic transform (logit)

$$(1.3)\quad l_i^{(r)} = \log\frac{h_i^{(r)}}{1-h_i^{(r)}} = \log\frac{a_i^{(r)}}{n_r-a_i^{(r)}}$$

estimates

$$(1.4)\quad l_i^{(r)} \approx \log\theta^{(r)} + \log\varepsilon_i.$$

Now denote by $l_{\circ}^{(r)}$ the average - weighted somehow, or unweighted - of $l_1^{(r)},\ldots,l_k^{(r)}$, similarly by $l_i^{(\circ)}$ the average of $l_i^{(1)},\ldots,l_i^{(k)}$, and finally by $l_{\circ}^{(\circ)}$ the total average of all the quantities $l_i^{(r)}$. (1.4) then leads to the results

$$(1.5)\quad \begin{cases} l_{\circ}^{(r)} \approx \log\theta^{(r)} + \log\bar{\varepsilon}, \\ l_i^{(1)} \approx \log\theta + \log\varepsilon_i, \\ l_{\circ}^{(1)} \approx \log\bar{\theta} + \log\bar{\varepsilon} \end{cases}$$

where $\bar{\varepsilon}$ and $\bar{\theta}$ denote the geometric means of the ε_i's and the $\theta^{(r)}$'s respectively.

Here, however, the situation has been somewhat idealized.

In practice we often meet cases where $a_i^{(r)}$ is either 0 or n_r so th
$l_i^{(r)}$ is $-\infty$ or $+\infty$. Therefore, before computing averages we have t
prepare the table of logits by eliminating rows and columns in suc
a way that the block left over comprises no infinities.

From (1.4) and (1.5) it is seen that

$$(1.6)\quad l_i^{(r)} - l_i^{(\cdot)} \approx \log \frac{\theta^{(r)}}{\bar{\theta}}$$

and

$$(1.7)\quad l_i^{(r)} - l_{\cdot}^{(r)} \approx \log \frac{\varepsilon_i}{\bar{\varepsilon}}.$$

In consequence we may at the same time check the model and estimate
the parameters by preparing two sets of diagrams, one set by plotti
for each r the logits $l_i^{(r)}$ against the averages $l_i^{(\cdot)}$, the other one b
plotting for each i the $l_i^{(r)}$ against the other set of averages $l_{\cdot}^{(r)}$.

If the model applies, the points in each diagram should
cluster around a straight line with unit slope, the position estimat
ing the parameters on the right in (1.6) or (1.7). These graphical
estimates we denote by

$$(1.8)\quad \overline{l}^{(r)} \approx \log \frac{\theta^{(r)}}{\bar{\theta}}$$

and

$$(1.9)\quad \overline{l}_i \approx \log \frac{\varepsilon_i}{\bar{\varepsilon}}.$$

Now the eliminated part of the logit table may be utilised.
We just have to plot the $l_i^{(r)}$'s for each i previously left out again
$\overline{l}^{(r)}$, and the other way round for each omitted r, marking the $+\infty$'s
and $-\infty$'s with arrows upwards and downwards. In this manner we shoul
according to the model obtain further straight lines with unit slope,
yielding supplementary estimates $\overline{l}^{(r)}$ and $\overline{l}_i$. The four corners where
both r and i had been eliminated are easily included in the operation

2. Method II.

In the following sections we shall deviate somewhat from the notations in Studies I, chapt. X, in writing γ instead of δ. Thus

(2.1) $\gamma_r = \gamma_r(\varepsilon_1, \ldots, \varepsilon_k)$

stands for the elementary symmetrical function of degree r in the variables $\varepsilon_1, \ldots, \varepsilon_k$ and correspondingly

(2.2) $\gamma_{r-1}^{(i)} = \gamma_{r-1}(\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_k)$

is the elementary symmetrical function of degree r-1 in the same variables except ε_i.

According to formulae (6.18) through (6.20) of chapt. X we have

(2.3) $l_i^{(r)} \approx \log \dfrac{\gamma_{r-1}^{(i)}}{\gamma_r^{(i)}} + \log \varepsilon_i$,

which compared with (1.2) suggests that

(2.4) $\log \Theta^{(r)} \simeq \log \dfrac{\gamma_{r-1}^{(i)}}{\gamma_r^{(i)}}$.

Thus the right hand term ought to be almost independent of i and within the simulation program it should be compared with averages of the Θ's for "individuals" with r in common.

For a numerical study of the said term the polynomials (2.2) should be computed from the given ε's. For that purpose it seems convenient first to compute the power sums

(2.5) $s_p = \sum_{j=1}^{k} \varepsilon_j^p$, $p = 0, 1, 2, \ldots$,

from them derive the γ_r's by means of the recurrence formula

(2.6) $r\gamma_r = \sum_{p=1}^{r} (-1)^{p-1} s_p \gamma_{r-p}$, $\gamma_0 = 1$, $\gamma_r = 0$ for $r>k$,

and finally apply the equations

(2.7) $\gamma_r^{(i)} = \gamma_r - \varepsilon_i \gamma_{r-1}^{(i)}$, $r = 1, 2, \ldots$,

starting from

(2.8) $\gamma_0^{(i)} = 1$.

3. Method III.

A slight modification in the application of the formulae (6.18) and (6.19) in chapt. X of the monograph leads to a clear cut separation of the variables referring to r and i.

In fact, from

$$(3.1)\quad h_i^{(r)} \approx \varepsilon_i \frac{\gamma_{r-1}^{(i)}}{\gamma_r}$$

and

$$(3.2)\quad 1-h_i^{(r-1)} \approx \frac{\gamma_{r-1}^{(i)}}{\gamma_{r-1}}$$

we may eliminate $\gamma_{r-1}^{(i)}$. Introducing the "quasi-logit"

$$(3.3)\quad k_i^{(r)} = \log \frac{h_i^{(r)}}{1-h_i^{(r-1)}}$$

we get

$$(3.4)\quad k_i^{(r)} \approx \log \varepsilon_i + \log \frac{\gamma_{r-1}}{\gamma_r}$$

from which $\log \varepsilon_i$ and $\log \frac{\gamma_{r-1}}{\gamma_r}$ may be estimated by Method I.

From the estimates of $\log \frac{\gamma_{r-1}}{\gamma_r}$ we may derive direct estimat of $\log \gamma_r$ which again may be utilized in the distribution law

$$(3.5)\quad p\{a_{\nu .} = r\} = \frac{\theta_\nu^r \gamma_r(\varepsilon_1, \ldots, \varepsilon_k)}{\prod_{(i)} (1+\theta_\nu \varepsilon_i)}$$

(cf. chapt. X (5.5)).

4. Method IV.

This method is based upon the empirical finding that a linear relation holds between $l_i^{(\circ)}$ and the estimated $\log \varepsilon_i$, viz.

$$(4.1)\quad l_i^{(\circ)} \approx 1{,}3 \log \varepsilon_i .$$

Taken together with the relation from Method I:

$$(4.2)\quad l_i^{(\circ)} \approx \log \bar{\theta} + \log \varepsilon_i ,$$

this would mean that

$$(4.3)\quad \log \bar{\theta} \approx 0{,}3 \log \varepsilon_i .$$

The content of this apparently paradoxical equation is investigated in the following.

In order to formulate the problem in more mathematical terms we consider the mean value $\mathcal{M}\{a_{\nu i}\}$ of the score of the ν'th person in the i'th test. According to the model it equals

$$(4.4)\quad \mathcal{M}\{a_{\nu i}\} = \frac{\theta_\nu \varepsilon_i}{1+\theta_\nu \varepsilon_i}$$ = the probability of answering correctly where

θ is the "ability" of the person, and ε_i is the "easiness" of the test. Defining $h_i^{(\circ)}$ as

$$(4.5)\quad h_i^{(\circ)} = \frac{a_{\cdot i}}{n}$$ (= relative raw score in i'th test)

we have

$$(4.6)\quad \mathcal{M}\{h_i^{(\circ)}\} = \frac{1}{n}\sum_{\nu=1}^{n} \mathcal{M}\{a_{\nu i}\} = \frac{1}{n}\sum_{\nu=1}^{n} \frac{\theta_\nu \varepsilon_i}{1+\theta_\nu \varepsilon_i} .$$

This mean value, however, is not to be understood as a mean value where the persons have been given the same set of weights for all values of i. To bring this point to the fore we shall write the mean value $\mathcal{M}\{h_i^{(\circ)}\}$ in the integral form:

$$(4.7)\quad \mathcal{M}\{h_i^{(\circ)}\} = \int_0^\infty \frac{\theta \varepsilon_i}{1+\theta \varepsilon_i}\, dH(\theta) ,$$

where $H(\theta)$ is the cumulated distribution function in the actual collection of persons tested.

From the mean value theorem of integral calculus it now follows that

$$(4.8)\quad \int_0^\infty \frac{\theta\varepsilon_i}{1+\theta\varepsilon_i}\, dH(\theta) = \frac{\bar{\theta}\varepsilon_i}{1+\bar{\theta}\varepsilon_i},$$

where $\bar{\theta}$ is a mean value of θ, depending, however, on ε_i.

Emperically we found this mean value to be a power of ε_i, and so we may ask if this is a theoretical possibility. This leads to the problem: Which functions $H(\theta)$ satisfy the integral equation

$$(4.9)\quad \int_0^\infty \frac{\theta\varepsilon}{1+\theta\varepsilon} dH(\theta) = \frac{\alpha\varepsilon^\beta}{1+\alpha\varepsilon^\beta}\ ?$$

The solution of this equation is deferred to sect. 5.

In case that the linearity between $\log\bar{\theta}$ and $\log \varepsilon_i$ holds, we may check the model by plotting $l_i^{(r)}$ against $l_i^{(o)}$, seeing for each r if the points lie about a straight line with a fixed slope $\neq 1$.

Combining the relations of sect. 1

$$(4.10)\quad l_i^{(r)} \approx \log \theta^{(r)} + \log \varepsilon_i$$

$$l_i^{(o)} \approx \log\bar{\theta} + \log \varepsilon_i$$

and the hypothesis

$$(4.11)\quad \log \bar{\theta} \simeq (\beta-1) \log \varepsilon_i$$

we find that

$$(4.12)\quad l_i^{(r)} - \frac{1}{\beta}\, l_i^{(o)} \approx \log \bar{\theta}^{(r)},$$

i.e. that the reciprocal value of the slope fof the straight line is for any r equal to the β in the integral equation. This β, by the way must depend on the total group of persons under consideration.

When testing (4.12) we shall split the question into two parts:

1. Does the linearity between $l_i^{(r)}$ and $l_i^{(o)}$ hold for each r?
2. Is the slope the same for all r's?

In order to do so by means of the computing machine we need a numerica substitute for the obvious graphical control. Provided that the number of problems is large (50-100), the following procedure is suggested:

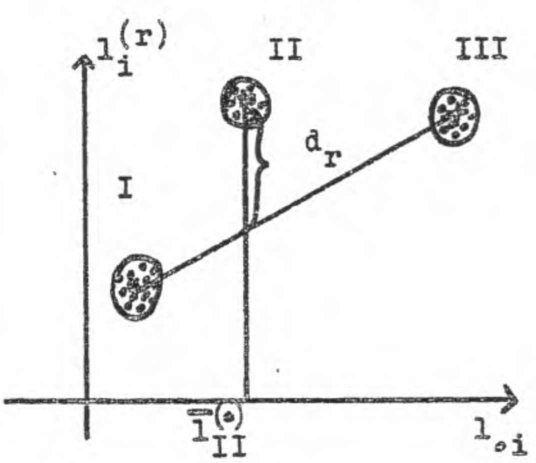
For any fixed r 3 sets of items are selected:

I. 10 items with finite $l_i^{(r)}$'s and for which the total $l_{\cdot i}$'s are as small as possible.

III. 10 items with finite $l_i^{(r)}$'s and for which the total $l_{\cdot i}$'s are as large as possible.

II. 10 items somewhere in between.

For each set the average $l_{\cdot i}$ is computed - $\bar{l}_I$, $\bar{l}_{II}$, and $\bar{l}_{III}$, say - and the average $\bar{l}_{\cdot i}^{(r)}$ as well - $\bar{l}_I^{(r)}$, $l_{II}^{(r)}$, and $\bar{l}_{III}^{(r)}$, say.



The slope of the line is then computed from the extreme groups I and III as

$$(4.13)\quad b_r = \frac{\bar{l}_{III}^{(r)} - \bar{l}_I^{(r)}}{\bar{l}_{III} - \bar{l}_I}.$$

Next we may calculate the distance of the gravity center $(\bar{l}_{II}, \bar{l}_{II}^{(r)})$ of group II from the line thus determined, the equation of which becomes

$$(4.14)\quad y - \bar{l}_I^{(r)} = b_r\,(x - \bar{l}_I),$$

the distance thus being

$$(4.15)\quad d_r = \bar{l}_{II}^{(r)} - \bar{l}_I^{(r)} - b_r(\bar{l}_{II}^{(r)} - \bar{l}_I^{(r)}).$$

In the formulae for the variances of b_r and d_r we shall neglect the variation of $\bar{l}_I^{(\circ)}$, $\bar{l}_{II}^{(\circ)}$, and $\bar{l}_{III}^{(\circ)}$, thus obtaining

$$(4.16)\quad V\{b_r\} = \frac{V\{\bar{l}_{III}^{(r)}\} - V\{\bar{l}_I^{(r)}\}}{(\bar{l}_{III}^{(\circ)} - \bar{l}_I^{(\circ)})^2}$$

and

$$(4.17)\quad V\{d_r\} = V\{\bar{I}_{II}^{(r)}\} + V\{\bar{I}_{I}^{(r)}\} + V\{b_r\}(\bar{I}_{II}^{(\cdot)} - \bar{I}_{I}^{(\cdot)})^2,$$

in which we have to insert the variances for the 3 ordinates of the gravity centers

$$(4.18)\quad V\{\bar{I}_j^{(r)}\} = \frac{1}{10}\,\frac{1}{\bar{h}_j^{(r)}(1-\bar{h}_j^{(r)})},\ j = I,\ II,\ III.$$

In case of linearity for all r's the quantities

$$(4.19)\quad u_r = \frac{d_r}{\sqrt{V\{d_r\}}}$$

should approximately conform to the u-distribution. At this stage th average, u, and the sum of squares SS_u may be helpful.

In order to check on the identity of the slopes β_r we may apply a variance-ratio test to the b_r's. However, it may be more informative to investigate the variation of b_r with r. In any case we need the weighted average

$$(4.20)\quad \bar{b} = \frac{\sum b_r/V\{b_r\}}{\sum 1/V\{b_r\}}$$

and the normalized deviations

$$(4.21)\quad v_r = \frac{b_r - \bar{b}}{\sqrt{V\{b_r\}}}$$

as well as the sum of squares

$$(4.22)\quad v^2 = \sum v_r^2.$$

Thus the answer to our questions requires the following two tables from the computing machine:

(4.23)

r	b_r	$\sqrt{V\{b_r\}}$	v_r
	$\bar{b}$		v^2

(4.24)

r	d_r	$\sqrt{V\{d_r\}}$	u_r
			$\bar{u}$ SS_u

5. Solution of an integral equation.

Substituting

$$(5.1)\quad \theta = e^{t},\ \varepsilon = e^{x},\ \beta = \frac{1}{\sigma},\ \alpha = e^{-\frac{\xi}{\sigma}},\ dH(\theta) = k(t)dt$$

in the integral equation

$$(5.2)\quad \int_{0}^{\infty} \frac{\theta\varepsilon}{1+\theta\varepsilon}\, dH(\theta) = \frac{\alpha\varepsilon^{\beta}}{1+\alpha\varepsilon^{\beta}}$$

we get

$$(5.3)\quad \int_{-\infty}^{\infty} \frac{e^{x+t}}{1+e^{x+t}}\, k(t)dt = \frac{\alpha e^{\beta x}}{1+\alpha e^{\beta x}} = \frac{e^{\frac{x-\xi}{\sigma}}}{1+e^{\frac{x-\xi}{\sigma}}}.$$

This equation may be solved by means of a LAPLACE-transformation. For $0 < \operatorname{Re} u < \frac{1}{\sigma}$ we have:

$$(5.4)\quad \int_{-\infty}^{\infty} e^{-ux}\, \frac{e^{\frac{x-\xi}{\sigma}}}{1+e^{\frac{x-\xi}{\sigma}}}\, dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} e^{-ux}\, \frac{e^{x+t}k(t)}{1+e^{x+t}}\, dt = \int_{-\infty}^{\infty} k(t)dt \int_{-\infty}^{\infty} \frac{e^{x+t}e^{-ux}}{1+e^{x+t}}\, dx$$

$$= \int_{-\infty}^{\infty} e^{ut}k(t)dt \int_{-\infty}^{\infty} \frac{e^{(1-u)x}}{1+e^{x}}\, dx.$$

Now

$$(5.5)\quad \int_{-\infty}^{\infty} \frac{e^{(1-u)x}}{1+e^{x}}\, dx = \frac{\pi}{\sin \pi u}$$

and accordingly

$$(5.6)\quad \int_{-\infty}^{\infty} e^{-ux}\, \frac{e^{\frac{x-\xi}{\sigma}}}{1+e^{\frac{x-\xi}{\sigma}}}\, dx = \sigma e^{-u\xi}\, \frac{\pi}{\sin \pi u\sigma}$$

which inserted into (5.4) gives

$$(5.7)\quad \int_{-\infty}^{\infty} e^{ut}k(t)dt = \sigma e^{-u\xi}\, \frac{\sin \pi u}{\sin \pi u\sigma},\quad -\frac{1}{\sigma} < \operatorname{Re} u < \frac{1}{\sigma}.$$

To this equation we may apply an inversion formula generalizing that of LEVY:

From

$$(5.8)\quad \int_{-\infty}^{\infty} e^{ut} p\{t\} dt = \varphi(u)$$

it follows that

$$(5.9)\quad p\{t\} = \lim_{\varepsilon\to 0,\, T\to\infty} \frac{1}{2\pi i}\left(\int_{-iT}^{-i\varepsilon}+\int_{i\varepsilon}^{iT}\right) e^{-tu}\varphi(u)du,$$

provided that this limit exists.

The result is that

$$(5.10)\quad k(t) = \frac{1}{\pi}\,\frac{e^{\frac{t+\xi}{\sigma}}\sin\frac{\pi}{\sigma}}{e^{2\frac{t+\xi}{\sigma}}+1+e^{\frac{t+\xi}{\sigma}}\cdot 2\cos\frac{\pi}{\sigma}} = \frac{1}{2\pi}\,\frac{\sin\frac{\pi}{\sigma}}{\cos h\frac{t+\xi}{\sigma}+\cos\frac{\pi}{\sigma}}$$

In order to prove this we first simplify (5.7) by putting

$$(5.11)\quad f(t) = k(t+\xi).$$

We then have to show that the solution of the equation

$$(5.12)\quad \int_{-\infty}^{\infty} e^{ut} f(t)dt = \frac{\sin\pi u}{\sin\pi\sigma u},\ 1<\sigma<\infty, -\frac{1}{\sigma}<\mathrm{Re}\ u<\frac{1}{\sigma}$$

is

$$(5.13)\quad f(t) = \frac{1}{\pi}\,\frac{e^{\frac{t}{\sigma}}\sin\frac{\pi}{\sigma}}{e^{2\frac{t}{\sigma}}+1+e^{\frac{t}{\sigma}}2\cos\frac{\pi}{\sigma}}.$$

According to the inversion formula

$$(5.14)\quad f(t) = \frac{\sigma}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \frac{\sin\pi u}{\sin\pi\sigma u}\, e^{-ut}du, -\frac{1}{\sigma}<\mathrm{Re}\ u<\frac{1}{\sigma},$$

which integral is absolutely and uniformly convergent for $-\infty<t<\infty$.

In the poles

$$(5.15)\quad u = \frac{p}{\sigma},\ p = \pm 1, \pm 2, \ldots$$

the integrand has the residues

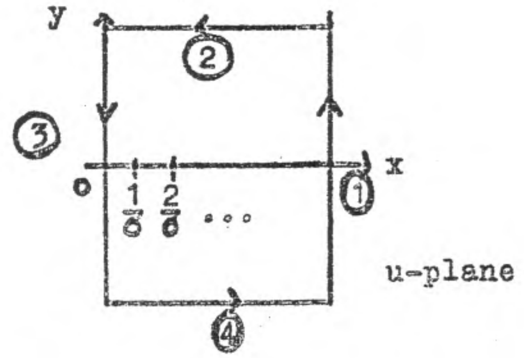
$$(5.16)\quad \left(\frac{\sigma\cdot\sin\pi u\cdot e^{-ut}}{\pi\sigma\cos\pi\sigma u}\right)_{u=\frac{p}{\sigma}} = \frac{(-1)^p}{\pi}\cdot\sin\frac{\pi p}{\sigma}\cdot e^{-t\frac{p}{\sigma}}$$

$$= \frac{(-1)^p}{2\pi i}\left(e^{(\pi i-t)\frac{p}{\sigma}} - e^{(-\pi i-t)\frac{p}{\sigma}}\right).$$

As $f(t)$ is an even function and continuous in $t = 0$, we nee only consider the case $t>0$.

For large numerical values of $u = x+iy$ the order of magnitude of the integrand becomes

$$(5.17)\quad \left|\frac{\sin \pi u}{\sin \pi \sigma u}\, e^{-ut}\right| \simeq \frac{e^{\pi|y|}}{e^{\pi\sigma|y|}}\, e^{-xt} = e^{\pi(1-\sigma)|y|-xt}.$$



It follows that the integrals along the paths 1 , 2 , and 4 indicated in the figure tend to 0 as $x = \frac{p+\frac{1}{2}}{\sigma} \to \infty$ and $|y| \to \infty$. In consequence $-f(t)$ equals the sum of the residues (5.16) taken over $p = 1, 2, \ldots$, i.e.

$$(5.18)\quad f(t) = -\sum_{p=1}^{\infty} \frac{(-1)^p}{2\pi i}\left(e^{(\pi i-t)\frac{p}{\sigma}} - e^{(-\pi i-t)\frac{p}{\sigma}}\right)$$

$$= \frac{-1}{2\pi i}\left(\frac{-e^{(\pi i-t)\frac{1}{\sigma}}}{1+e^{(\pi i-t)\frac{1}{\sigma}}} - \frac{-e^{(-\pi i-t)\frac{1}{\sigma}}}{1+e^{(-\pi i-t)\frac{1}{\sigma}}}\right)$$

from which (5.13) follows immediately.

The cumulated distribution function is

$$(5.19)\quad F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{\sin\frac{\pi}{\sigma}}{e^{\frac{t}{\sigma}}+e^{-\frac{t}{\sigma}}+2\cos\frac{\pi}{\sigma}}\,dt = \frac{\sigma}{\pi}\,\mathrm{arc\ tg}\,\frac{\sin\frac{\pi}{\sigma}}{e^{-\frac{x}{\sigma}}+\cos\frac{\pi}{\sigma}}.$$

6. Method V.

This method is based upon the following lemma: If $x_1, \ldots, x_n$ are independent variables and we set

$$(6.1) \quad \begin{aligned} y_1 &= x_1 \\ y_2 &= x_1 + x_2 \\ y_\nu &= x_1 + x_2 + \ldots x_\nu \\ y &= x_1 + x_2 + \ldots x_n \end{aligned}$$

then

$$(6.2) \quad p\{y_1, \ldots, y_n\} = \prod_{\nu=1}^{n} p\{x_\nu | y_\nu\} p\{y_n\}.$$

Proof: To the identity

$$(6.3) \quad p\{y_1, \ldots, y_n\} = p\{y_1 | y_2, \ldots, y_n\} p\{y_2 | y_3, \ldots, y_n\}, \ldots, p\{y_{n-1} | y_n\} p|$$

we apply the relation

$$(6.4) \quad p\{y_{n-2} | y_{n-1}, y_n\} = \frac{p\{y_{n-2}, y_{n-1}, y_n\}}{p\{y_{n-1}, y_n\}} = \frac{p\{y_{n-2}, y_{n-1}, x_n\}}{p\{y_{n-1}, x_n\}}$$

$$= \frac{p\{y_{n-2}, y_{n-1}\}}{p\{y_{n-1}\}} = p\{y_{n-2} | y_{n-1}\}$$

and its analogues

$$(6.5) \quad p\{y_{n-3} | y_{n-2}, y_{n-1}, y_n\} = p\{y_{n-3} | y_{n-2}\}, \text{ etc.,}$$

thus obtaining

$$(6.6) \quad p\{y_1, \ldots, y_n\} = p\{y_1 | y_2\} p\{y_2 | y_3\} \ldots p\{y_{n-1} | y_n\} p\{y_n\}.$$

Furthermore

$$(6.7) \quad p\{y_{\nu-1} | y_\nu\} = \frac{p\{y_{\nu-1}, y_\nu\}}{p\{y_\nu\}} = \frac{p\{x_\nu, y_\nu\}}{p\{y_\nu\}} = p\{x_\nu | y_\nu\},$$

which immediately leads to the desired relation

(6.8) $p\{x_1,\dots,x_n\} = p\{y_1,\dots,y_n\} = p\{x_1|y_1\}p\{x_2|y_2\}\dots p\{x_n|y_n\}p\{y_n\}.$

Now to use this theorem to astimate the ε_i, we consider the matrix

(6.9) $$A = (a_{\nu i})$$

of the scores of the persons $\nu = 1,\dots,n$ in the k test problems. For each row $a_{\nu i},\dots,a_{\nu k}$ in this matrix, we define the cumulated numbers of correct answers

(6.10) $$\begin{aligned} c_{\nu 1} &= a_{\nu 1} \\ c_{\nu 2} &= a_{\nu 1} + a_{\nu 2} \\ &\vdots \\ c_{\nu k} &= a_{\nu 1} + a_{\nu 2} + \dots + a_{\nu k} = a_{\nu .} \end{aligned}$$

From the lemma it follows that

(6.11) $$p\{a_{\nu 1},\dots,a_{\nu k}\} = \prod_{i=1}^{k} p\{a_{\nu i}|c_{\nu i}\}p\{c_{\nu k}\} = \prod_{i=1}^{k} p\{a_{\nu i}|c_{\nu i}\}p\{a_{\nu .}\}$$

and by multiplication over ν we get

(6.12) $$p\{A\} = \prod_{\nu=1}^{n}\prod_{i=1}^{k}(p\{a_{\nu i}|c_{\nu i}\}p\{c_{\nu k}\}).$$

Now

(6.13) $$\prod_{\nu=1}^{n} p\{c_{\nu k}\} = \prod_{\nu=1}^{n} p\{a_{\nu .}\} = p\{A_{*.}\},$$

and therefore

(6.14) $$p\{A|A_{*.}\} = \prod_{\nu=1}^{n}\prod_{i=1}^{k} p\{a_{\nu i}|c_{\nu i}\}.$$

With a slight alteration in notation the formulae (5.3) and (5.5) in chapt. X of the monograph are:

(6.15) $$p\{a_{\nu 1},\dots,a_{\nu k}\} = \frac{\theta_\nu^{a_{\nu .}}\,\varepsilon_1^{a_{\nu 1}},\dots,\varepsilon_k^{a_{\nu k}}}{\prod_{(i)}(1+\theta_\nu\varepsilon_i)}$$

(6.16) $$p\{a\quad\} = \frac{\theta_\nu^{a_{\nu .}}\gamma_{a_{\nu .}}(\varepsilon_1,\dots,\varepsilon_k)}{\prod_{i}(1+\theta_\nu\varepsilon_i)},$$

where $\gamma_{a_{\nu .}}(\varepsilon_1,\dots,\varepsilon_k)$ is the symmetric polynomial of degree $a_{\nu .}$ in the variables $\varepsilon_1,\ldots,\varepsilon_k$. Dividing (6.15) into (6.16) we get

$$(6.17)\quad p\{a_{\nu 1},\ldots,a_{\nu k}|a_{\nu .}\} = \frac{\varepsilon_1^{a_{\nu 1}}\cdots\varepsilon_k^{a_{\nu k}}}{\gamma_{a_{\nu .}}(\varepsilon_1,\ldots,\varepsilon_k)},$$

and summing over all possible combinations $a_{\nu 1},\ldots,a_{\nu k}$ with a fixed total $a_{\nu .}$ we find the two supplementary formulae:

$$(6.18)\quad p\{a_{\nu i} = 1|a_{\nu .}\} = \frac{\varepsilon_i\gamma_{a_{\nu .}-1}(\varepsilon_1,\ldots,\varepsilon_{i-1},\varepsilon_{i+1},\ldots,\varepsilon_k)}{\gamma_{a_{\nu .}}(\varepsilon_1,\ldots,\varepsilon_k)},$$

$$p\{a_{\nu i} = 0|a_{\nu .}\} = \frac{\gamma_{a_{\nu .}}(\varepsilon_1,\ldots,\varepsilon_{i-1},\varepsilon_{i+1},\ldots,\varepsilon_k)}{\gamma_{a_{\nu .}}(\varepsilon_1,\ldots,\varepsilon_k)}.$$

Applying them to the first i test problems only we may write $c_{\nu i}$ for $a_{\nu .}$ and get

$$(6.19)\quad p\{a_{\nu i} = 1|c_{\nu i}\} = \frac{\varepsilon_i\gamma_{c_{\nu i}-1}(\varepsilon_1,\ldots,\varepsilon_{i-1})}{\gamma_{c_{\nu i}}(\varepsilon_1,\ldots,\varepsilon_i)}$$

$$p\{a_{\nu i} = 0|c_{\nu i}\} = \frac{\gamma_{c_{\nu i}-1}(\varepsilon_1,\ldots,\varepsilon_{i-1})}{\gamma_{c_{\nu i}}(\varepsilon_1,\ldots,\varepsilon_i)}.$$

For short we shall write $\boldsymbol{\varepsilon}_{i-1}$ and $\boldsymbol{\varepsilon}_i$ for the "vectors" $(\varepsilon_1,\ldots,\varepsilon_{i-1})$ and $(\varepsilon_1,\ldots,\varepsilon_i)$ so that the formulae become

$$(6.20)\quad p\{a_{\nu i} = 1|c_{\nu i}\} = \frac{\varepsilon_i\gamma_{c_{\nu i}-1}(\boldsymbol{\varepsilon}_{i-1})}{\gamma_{c_{\nu i}}(\boldsymbol{\varepsilon}_i)},$$

$$p\{a_{\nu i} = 0|c_{\nu i}\} = \frac{\gamma_{c_{\nu i}}(\boldsymbol{\varepsilon}_{i-1})}{\gamma_{c_{\nu i}}(\boldsymbol{\varepsilon}_i)}.$$

Now consider the persons which have a fixed $c_{\nu i} = c\ (\leq i)$. Denote by $a_{.i}^{(c)}$ and $b_{.i}^{(c)}$ the number of them for which $a_{\nu i} = 1$ and $a_{\nu i} = 0$, respectively, and by $n_{.i}^{(c)}$ the total number of them:

$$(6.21)\quad n_{.i}^{(c)} = a_{.i}^{(c)} + b_{.i}^{(c)}.$$

For this group of persons we have

$$(6.22)\quad p\{a_{\cdot\nu}^{(c)} \mid c_{\nu i} = c\} = \binom{n_{\cdot i}^{(c)}}{a_{\cdot i}^{(c)}} \frac{(\varepsilon_i \gamma_{c-1}(\mathcal{E}_{i-1}))^{a_{\cdot i}^{(c)}} \gamma_c(\mathcal{E}_{i-1})^{b_{\cdot i}^{(c)}}}{\gamma_c(\mathcal{E}_i)^{n_{\cdot i}^{(c)}}}$$

from which we obtain the estimates:

$$(6.23)\quad \frac{a_{\cdot i}^{(c)}}{n_{\cdot i}^{(c)}} \approx \frac{\varepsilon_i \gamma_{c-1}(\mathcal{E}_{i-1})}{\gamma_c(\mathcal{E}_i)} \text{ and}$$

$$(6.24)\quad \frac{b_{\cdot i}^{(c)}}{n_{\cdot i}^{(c)}} \approx \frac{\gamma_c(\mathcal{E}_{i-1})}{\gamma_c(\mathcal{E}_i)}.$$

There is some doubt left as to the stochastical independence for different i. In the following we shall assume it to be manifest.
Dividing (6.24) into (6.23) and taking the logarithm we get

$$(6.25)\quad \log \frac{a_{\cdot i}^{(c)}}{b_{\cdot i}^{(c)}} = l_{\cdot i}^{(c)} \approx \log \varepsilon_i + \log \frac{\gamma_{c-1}(\mathcal{E}_{i-1})}{\gamma_c(\mathcal{E}_{i-1})}$$

$$c = 1,\dots,i-1.$$

Summation over c yields

$$(6.26)\quad \sum_{c=1}^{i-1} l_i^{(c)} \approx i \log \varepsilon_i - \sum_{j=1}^{i} \log \varepsilon_j.$$

For i-1 this equation becomes

$$(6.27)\quad \sum_{c=1}^{i-2} l_{\cdot i-1}^{(c)} \approx (i-1) \log \varepsilon_{i-1} - \sum_{j=1}^{i-1} \log \varepsilon_j,$$

and by subtracting the two equations we obtain

$$(6.28)\quad \sum_{c=1}^{i-1} l_{\cdot i}^{(c)} - \sum_{c=1}^{i-2} l_{\cdot i-1}^{(c)} \approx (i-1) \log \frac{\varepsilon_i}{\varepsilon_{i-1}}$$

or

$$(6.29)\quad \frac{1}{i-1} \sum_{c=1}^{i-1} l_{\cdot i}^{(c)} - \sum_{c=1}^{i-2} l_{\cdot i-1}^{(c)} \approx \frac{\varepsilon_i}{\varepsilon_{i-1}}.$$

Estimates of the ε_i's may now be obtained by successive additions.

Putting

$$(6.30)\quad k_{\cdot i}^{(c)} = \log \frac{h_{\cdot i}^{(c)}}{1-h_{\cdot i}^{(c-1)}}$$

we have

$$(6.31)\quad k_{\cdot i}^{(c)} \approx \log \varepsilon_i + \log \frac{\gamma_{c-1}(\varepsilon_i)}{\gamma_c(\varepsilon_i)}$$

and consequently

$$(6.32)\quad d_{\cdot i}^{(c)} = l_{\cdot i}^{(c)} - k_{\cdot i-1}^{(c)} \approx \log \frac{\varepsilon_i}{\varepsilon_{i-1}}$$

which provides a new estimate of the ε_i's.

To control the model we may plot $l_{\cdot i}^{(c)}$ against $k_{\cdot i-1}^{(c)}$ for fixed i and variable c. If the model applies, the points will cluster around a straight line with unit slope.