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MODELS FOR DESCRIPTION OF THE TIME-SPACE DISTRIBUTION OF TRAFFIC ACCIDENTS

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Methods of statistical analysis

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INTRODUCTION

The deliberations contained in this report refer to an analysis carried into effect with a view to determining whether the periodic speed limits introduced in 1961-64 may in any way be shown to have affected the traffic accident statistics in Denmark. The results of the mentioned analysis are set forth in (4). As basic material for the analysis was used the number of motor vehicle accidents involving personal injury, distributed on certain dates throughout the years under review, and also on road types. By road type is meant that the overall Danish road network was subdivided according to the classification of the respective roads (trunk roads, highways, secondary roads), the cross-section of the particular road (the width and number of lanes), and the character of the marginal developments.

Reference is also made in this report to an official Swedish report on the same subject (5).

§ 1. The Multiplicative Poisson Law

The above law was first used in connection with an evaluation of related attainment tests in oral reading but has proved fruitful since then in different other fields.

It is applied in an effort simultaneously to describe simple structural correlations and stochastic variations of positive integers (≥ 0), divided according to two different but intersecting criteria. In respect of the problem discussed in this report let us imagine that the time (t) intersects the road type (i), the observation a_{it} being the number of accidents of a given type (in the case at issue, motor vehicle accidents causing personal injury).

Primarily, an attempt is made to consider the accident flow an elementary stochastic process with the parameter λ_{it}, which in the given situation - the intensity of the process on the section of the road network under investigation, integrated over the period of time, e.g. 24 hours, denoted by t - in some way or other is determined by the combination of i and t:

$$(1.1) \qquad p\{a_{it}\} = e^{-\lambda_{it}} \cdot \frac{\lambda_{it}^{a_{it}}}{a_{it}!} \quad .$$

The decisive point is, however, that it is considered possible to describe this interdependency as the product of two factors, one of which, ξ_i, characterizes i (the road type) at any time, while the second one, η_t, characterizes t (the "point" in time) irrespective of road type:

$$(1.2) \qquad \lambda_{it} = \xi_i \eta_t \; .$$

Equations (1.1) and (1.2) are definitions of the Multiplicative Poisson Law. Concerning the theory underlying this distribution and the various modes of application the reader is mainly referred to (1), Chapt. II, III, VIII and IX and to (2), Målingsmodeller I, pp. 1-11. With a view to adapting important results of a more recent date (3) to our purposes, we shall, however, first set forth a special formulation of a wellknown theorem on Poisson distributions (1), pp. 129-130, followed by its converse, viz.:

Theorem 1: <u>If the stochastically independent variables a and b are Poisson distributed, with the parameters $\alpha\xi$ and $\beta\xi$, the sum of these variables</u>

$$c = a+b \tag{1.3}$$

<u>will be Poisson distributed, with the parameter $(\alpha+\beta)\xi$, and the distribution of a and b conditional on a constant sum c will be binomial and will have the parameters c and $\frac{\alpha}{\alpha+\beta}$</u> :

$$p\{a|c\} = \binom{c}{a} \frac{\alpha^a \ \beta^b}{(\alpha+\beta)^c} . \tag{1.4}$$

The first part of this theorem is the addition formula for the Poisson distribution and derives immediately from the fact that a summation over a+b = c of

$$p\{a,b\} = e^{-\xi(\alpha+\beta)} \cdot \frac{\xi^c}{c!} \cdot \frac{c!}{a!b!} \cdot \alpha^a \ \beta^b \tag{1.5}$$

yields

$$p\{c\} = e^{-\xi(\alpha+\beta)} \frac{\xi^c}{c!} \cdot (\alpha+\beta)^c . \tag{1.6}$$

Thereafter, the conditional distribution (1.4) is arrived at by dividing (1.6) into the simultaneous distribution (1.5).

The converse of Theorem 1, which may be ascribed to S.D. Chatterji (3), reads as follows:

Theorem 2: <u>If the distribution of the stochastically independent variables a and b, conditioned by their sum c, is binomial (1.4), with the same probabilistic parameter for any integral value c > 0, then the marginal distributions of a, b, and c will follow the Poisson Law in every imaginable situation, with parameters of the form of $\alpha\xi$, $\beta\xi$ and $(\alpha+\beta)\xi$, ξ being a parameter which is characteristic of the respective situation.</u>

The proof is conducted through an examination of which distribution laws $p\{a\}$ and $p\{b\}$ have the mean value and the variance of the conditional distribution $p\{a,b|c\}$ in common with (1.4). As an instrument we use probability generating functions

$$\Pi\{{}^{a}_{x}\} = \sum p\{a\}\, x^{a} \ . \tag{1.7}$$

As will be known, these functions are subject to the following rules:

If a and b are stochastically independent,

$$\Pi\{{}^{a,b}_{x,y}\} = \Pi\{{}^{a}_{x}\}\Pi\{{}^{b}_{y}\} \tag{1.8}$$

Irrespective whether a and b are stochastically independent, the generating function for their sum will be

$$\Pi\{{}^{a+b}_{x}\} = \Pi\{{}^{a,b}_{x,x}\} \ . \tag{1.9}$$

The generating functions for the conditional probabilities $p\{a,b|c\}$, $c=a+b$ may be obtained by expanding $\Pi\{{}^{a,b}_{xz,yz}\}$ with respect to powers of z since

$$\Pi\{{}^{a,b}_{xz,yz}\} = \Sigma\Pi\{{}^{a,b}_{x,y}|c\}\ p\{c\}z^{c} \ , \tag{1.10}$$

the left hand side of this equation having the form of (1.8).

If we now differentiate

$$\Pi\{{}^{a}_{xz}\}\ \Pi\{{}^{b}_{yz}\} = \Sigma\Pi\{{}^{a,b}_{x,y}|c\}\ p\{c\}z^{c} \tag{1.11}$$

in respect of x and thereafter make $x = y = 1$, we will have

$$\left(\frac{\partial\Pi\{{}^{a,b}_{x,y}|c\}}{\partial x}\right)_{x=y=1} = \mathcal{M}\{a|c\} = \frac{\alpha}{\alpha+\beta}\cdot c \tag{1.12}$$

yielding also

$$z\Pi'\{{}^{a}_{z}\}\ \Pi\{{}^{b}_{z}\} = \frac{\alpha}{\alpha+\beta}\ \Sigma c p\{c\}z^{c} = \frac{\alpha}{\alpha+\beta}\ z\Pi'\{{}^{c}_{z}\}. \tag{1.13}$$

When combined with (1.9) the result is

$$(1.14) \qquad \frac{\Pi'\{\frac{a}{z}\}}{\Pi\{\frac{a}{z}\}} = \frac{\alpha}{\alpha+\beta} \frac{\Pi'\{\frac{c}{z}\}}{\Pi\{\frac{c}{z}\}} ,$$

which by integration gives

$$(1.15) \qquad \Pi\{\tfrac{a}{z}\} = (\Pi\{\tfrac{c}{z}\})^{\frac{\alpha}{\alpha+\beta}} .$$

Analogous to the above

$$(1.14a) \qquad \frac{\Pi'\{\frac{b}{z}\}}{\Pi\{\frac{b}{z}\}} = \frac{\beta}{\alpha+\beta} \frac{\Pi'\{\frac{c}{z}\}}{\Pi\{\frac{c}{z}\}}$$

and

$$(1.15a) \qquad \Pi\{\tfrac{b}{z}\} = (\Pi\{\tfrac{c}{z}\})^{\frac{\beta}{\alpha+\beta}}$$

Subsequently (1.11) is differentiated in respect of both x and y prior to making them = 1. This means that

$$(1.16) \qquad \left(\frac{\partial^2 \Pi\{\frac{a,b}{x,y}|c\}}{\partial x \, \partial y}\right)_{x=y=1} = \mathcal{M}\{ab|c\}$$

$$= \mathcal{M}\{a|c\} \, \mathcal{M}\{b|c\} - \mathcal{D}\{a|c\}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2} \cdot c(c-1)$$

since

$$\mathcal{D}\{a,b|c\} = \mathcal{M}\{(a - \frac{\alpha c}{\alpha+\beta})(b - \frac{\beta c}{\alpha+\beta})|c\}$$

$$(1.17) \qquad = +\mathcal{M}\{(a - \frac{\alpha c}{\alpha+\beta})(b - (1 - \frac{\alpha}{\alpha+\beta})c)|c\}$$

$$= -\mathcal{D}\{a|c\} = -\frac{\alpha\beta}{(\alpha+\beta)^2} c .$$

Consequently,

$$(1.18) \qquad z^2 \Pi'\{\tfrac{a}{z}\} \, \Pi'\{\tfrac{b}{z}\} = \frac{\alpha\beta}{(\alpha+\beta)^2} = z^2 \Pi''\{\tfrac{c}{z}\}$$

and through division by $\Pi\{{c \atop z}\}$ and utilization of (1.14) and (1.14a) we arrive at

$$(1.19) \qquad \left(\frac{\Pi'\{{c \atop z}\}}{\Pi\{{c \atop z}\}}\right)^2 = \frac{\Pi''\{{c \atop z}\}}{\Pi\{{c \atop z}\}}$$

which in the first instance is integrated to

$$(1.20) \qquad \log\Pi\{{c \atop z}\} = \log\Pi'\{{c \atop z}\} - \log\mu$$

where

$$(1.21) \qquad \mu = \mathscr{M}\{c\} .$$

Taking the antilogarithm of (1.20), and integrating a second time, we obtain

$$(1.22) \qquad \Pi\{{c \atop z}\} = e^{\mu(z-1)} .$$

Consequently, in accordance with (1.15) and (1.15a)

$$(1.22a) \qquad \Pi\{{a \atop z}\} = e^{\frac{\alpha}{\alpha+\beta}\mu(z-1)}$$

and

$$(1.22b) \qquad \Pi\{{b \atop z}\} = e^{\frac{\beta}{\alpha+\beta}\mu(z-1)} ,$$

either of which gives a specific Poisson distribution. That the conditional distribution $p(a|c)$ is (1.4) may be inferred from Theorem 1.

If considering this result as an isolated case, no special importance may be attached to the fact that the parameters have one factor μ in common, but when taking into consideration that this reasoning applies to every single t (the date), and that the basic assumption is that the conditional distributions (1.4) have the same parameter, viz. $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$, respectively, for any possible t, then μ becomes a parameter which may vary at random with t, but now it is really a question of a decomposition of the Poisson parameters, one factor μ_t varying with the time but common to the two road types, while the second one, $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$, is independent of the time but specific for the road type.

The reasoning may be directly generalized to more than two road types. If $a_1,\ldots\ldots,a_k$, with the sum c, denote independent Poisson distributed variables, (1.4) is replaced by the polynomial distribution

$$(1.23) \qquad p(a_1,\ldots\ldots,a_k|c) = \binom{c}{a_1,\ldots,a_k} \frac{\xi_1^{a_1}\ldots\xi_k^{a_k}}{(\xi_1+\ldots+\xi_k)^c} ,$$

and based upon this formula the converse of the above theorem may be proved, e.g. by means of the binomial distributions applying to any a_i and $c-a_i$.

§ 2. Requirements for χ^2-tests for a comparison of accident distributions according to road type

By means of the converse theorem for the Polynomial Law it is possible to throw light on the requirements for a valid comparison to be made between the distributions of a specific type of accidents on various segments of the road network.

So far no investigations have been made to determine whether the distribution of accidents involving severe personal injuries according to dates and road types follows the multiplicative Poisson law as is the case with accidents involving minor personal injuries. The Swedish report (5) on the 1961/62 investigations, however, gives the distribution on road types during periods with and without introduction of speed limits. No statistical tests were performed in support of the difference claimed to exist when comparing the two distributions, but a χ^2-test would seem to be indicated and would answer the question whether, from a theoretical point of view, the distribution might possibly be the same for the two periods. Now both materials suffer from a high degree of inhomogeneity, both as regards external traffic conditions and the absolute number of accidents. This in itself actually does not eliminate the possibility of attaching any significance to the χ-test. In this connection, however, it would seem to be an absolute condition that the distribution on road types must be stable for each specific period regardless of the wide differences in external conditions and absolute figures. In accordance with the converse theorem of the multiplicative Poisson law this would, however, be tantamount to accepting the validity of the latter theorem, which, in fact, has not been determined. Data in support of such validity are, however, to be found in Tables 1-3 of the report.

The situation is exactly the same in respect of fatal accidents, but within this category no data of a corresponding nature have been published. The tables merely show <u>the number of persons killed</u>, quite a different matter than <u>the number of fatal accidents</u>.

§ 3. The Pascal-Polya Distribution

In (4, B § 5) we were led to the question whether it may be thought possible for two stochastically independent variables a and b to follow such distributions that the conditional distribution of (a,b) for the given sum

$$(3.1) \qquad c = a+b$$

may have a mean and a variance of the respective forms

$$(3.2) \qquad \mathcal{M}\{a|c\} = c\theta$$

and

$$(3.3) \qquad \mathcal{V}\{a|c\} = c\theta(1-\theta) \cdot \frac{\gamma+c}{\gamma+1}, \quad \gamma > 0 .$$

Like in § 1, this question is examined by means of generating functions, using as basis, in particular, the equation (1.11):

$$(3.4) \qquad \Pi\{{a \atop xz}\}\,\Pi\{{b \atop yz}\} = \Sigma\Pi\{{a,b \atop x,y}|c\}\,p\{c\}z^c .$$

By differentiation in respect of x and by making x = y = 1 we arrive, like in § 1, at

$$(3.5) \qquad \frac{\Pi'\{{a \atop z}\}}{\Pi\{{a \atop z}\}} = \theta \cdot \frac{\Pi'\{{c \atop z}\}}{\Pi\{{c \atop z}\}} ,$$

analogous to which

$$(3.5a) \qquad \frac{\Pi'\{{b \atop z}\}}{\Pi\{{b \atop z}\}} = (1-\theta) \cdot \frac{\Pi'\{{c \atop z}\}}{\Pi\{{c \atop z}\}} ,$$

from which it follows that

$$(3.6) \qquad \Pi\{{a \atop z}\} = (\Pi\{{c \atop z}\})^{\theta}$$

and

$$(3.6a) \qquad \Pi\{{b \atop z}\} = (\Pi\{{c \atop z}\})^{1-\theta} .$$

Subsequently, differentiation is made in respect of both x and y before assigning to them the value of = 1. Instead of (1.18) we will now have

$$(3.7) \qquad z^2 \Pi'\{{a \atop z}\} \; \Pi'\{{b \atop z}\} = \frac{\gamma}{\gamma+1} \, \theta(1-\theta) z^2 \; \Pi''\{{c \atop z}\} .$$

Through division by $z^2 \Pi\{{c \atop z}\}$, and applying (1.8) as well as (3.5) and (3.5a), we obtain

$$(3.8) \qquad \left(\frac{\Pi'\{{c \atop z}\}}{\Pi\{{c \atop z}\}}\right)^2 = \frac{\gamma}{\gamma+1} \cdot \frac{\Pi''\{{c \atop z}\}}{\Pi\{{c \atop z}\}}$$

with the solution

$$(3.9) \qquad \frac{\gamma}{\gamma+1} \left(\log\Pi'\{{c \atop z}\} - \log\mu\right) = \log\Pi\{{c \atop z}\} ,$$

the integration constant of which is

$$(3.10) \qquad \log\mu = \log\Pi'\{{c \atop 1}\} = \log\mathcal{M}\{c\} .$$

After rearrangement to

$$\frac{\Pi'\{{c \atop z}\}}{(\Pi\{{c \atop z}\})^{1+\gamma^{-1}}} = \mu$$

the equation is integrated to

$$(3.11) \qquad \Pi\{{c \atop z}\} = \left(1 - \frac{\mu}{\gamma}(z-1)\right)^{-\gamma}$$

which based on (3.6) and (3.6a) yields

$$(3.12) \qquad \Pi\{{a \atop z}\} = (1 - \frac{\mu}{\gamma}(z-1))^{-\alpha}$$

and

$$(3.12a) \qquad \Pi\{{b \atop z}\} = (1 - \frac{\mu}{\gamma}(z-1))^{-\beta} .$$

This, however, only serves to prove that if any such distributions of a and b exist that the mean and variance of the conditional probability of (a,b) at a given c = a+b are the same as (3.2) and (3.3), respectively, then their generating functions must be of the forms (3.12) and (3.12a).

This, however, does not even prove that these functions produce any probabilistic distributions at all. That this nevertheless is the fact may be concluded, though, from noting that the coefficients of the powers of z in the expansion of

$$(3.13) \qquad \Pi\{{a \atop z}\} = (1 - \frac{\mu}{\gamma+\mu} z)^{-\alpha} \cdot (\frac{\gamma}{\gamma+\mu})^{\alpha}$$

are positive:

$$(3.14) \qquad p\{a|\alpha, \frac{\mu}{\gamma}\} = (-1)^a \binom{-\alpha}{a} \cdot (\frac{\mu}{\gamma+\mu})^a \cdot (\frac{\gamma}{\gamma+\mu})^{\alpha} ,$$

and by analogy

$$(3.14a) \qquad p\{b|\beta, \frac{\mu}{\gamma}\} = (-1)^b \binom{-\beta}{b} (\frac{\mu}{\gamma+\mu})^b (\frac{\gamma}{\gamma+\mu})^{\beta} .$$

Furthermore, in accordance with (3.11)

$$(3.15) \qquad p\{c|\gamma, \frac{\mu}{\gamma}\} = (-1)^c \binom{-\gamma}{c} (\frac{\mu}{\gamma+\mu})^c (\frac{\mu}{\gamma+\mu})^{\gamma} ,$$

and from the above three distributions the conditional distribution of (a,b), given the sum a+b = c, is then found to be

$$(3.16) \qquad p\{a,b|\alpha,\beta, \frac{\mu}{\gamma}\} = \frac{(-1)^a \binom{-\alpha}{a}(-1)^b \binom{-\beta}{b}}{(-1)^c \binom{-\gamma}{c}}$$

$$= \frac{(\alpha,\alpha+a)}{a!} \cdot \frac{(\beta,\beta+b)}{b!} \cdot \frac{c!}{(\gamma,\gamma+c)}$$

which is independent of the parameter.

A variant of the distribution type (3.14) was first given by Pascal in a form which formed a contrast to the binomial distribution. Polya's name is also identified with this type of distribution, but more or less as an extension of the Poisson distribution, the parameter λ of which was considered to be a stochastic variable following a γ distribution. We will, therefore, call it the Pascal-Polya distribution and point out that if letting $\gamma \to \infty$ while μ and θ are kept constant, the Poisson distribution is obtained as a borderline case.

The interpretation of the parameters in the distributions thus developed requires a certain amount of deliberation.

μ, which is the mean of c, may vary according to time (day-week number), while α, β and γ as parameters in the conditional mean and variance are assumed to be constant. To elucidate their interrelationship we point out that the variables a and b refer to two specific years, i and j; instead of α and β we therefore now write α_i and α_j. γ is then a parameter which is characteristic of the <u>pair</u> (i,j). Our argumentation in the foregoing has been based on the assumption that γ need be constant only as long as this same pair is being maintained, while it may easily change when we change over to a different 2-year period or to other periods within the same two years. γ is, therefore, denoted as γ_{ij}.

We now revert to μ, which as mentioned may vary according to the day-week number t but may also very well be affected by the pair (i,j). Provisionally, therefore, we write μ_{ijt} instead of μ_t, and finally a, b, and c are denoted more precisely as a_{it}, a_{jt}, $c_t = a_{it} + b_{jt}$.

Using these notations, we have

$$(3.17) \qquad p\{a_{it}\} = (-1)^{a_{it}} \binom{-\alpha_i}{a_{it}} \left(\frac{\mu_{ijt}}{\gamma_{ij} + \mu_{ijt}}\right)^{a_{it}} \left(\frac{\gamma_{ij}}{\gamma_{ij} + \mu_{ijt}}\right)^{\alpha_i}$$

and

$$(3.18) \qquad p\{a_{jt}\} = (-1)^{a_{jt}} \binom{-\alpha_j}{a_{jt}} \left(\frac{\mu_{ijt}}{\gamma_{ij} + \mu_{ijt}}\right)^{a_{jt}} \left(\frac{\gamma_{ij}}{\gamma_{ij} + \mu_{ijt}}\right)^{\alpha_j} ,$$

from which it is derived that

$$p\{c_t\} = (-1)^{c_t} \binom{-\alpha_i - \alpha_j}{c_t} \left(\frac{\mu_{ijt}}{\gamma_{ij} + \mu_{ijt}}\right)^{c_t} \left(\frac{\gamma_{ij}}{\gamma_{ij} + \mu_{ijt}}\right)^{\alpha_i + \alpha_j} \tag{3.19}$$

and

$$p\{a_{it}, a_{jt} | c_t\} = \frac{(-1)^{a_{it}} \binom{-\alpha_i}{a_{it}} (-1)^{a_{jt}} \binom{-\alpha_j}{a_{jt}}}{(-1)^{c_t} \cdot \binom{-\alpha_i - \alpha_j}{c_t}} \tag{3.20}$$

The mean and variance of this conditional distribution are finally obtained by plain algebra, viz.

$$\mathcal{E}\{a_{it} | c_t\} = \frac{\alpha_i}{\alpha_i + \alpha_j} \cdot c_t \tag{3.21}$$

and

$$\mathcal{V}\{a_{it} | c_t\} = \frac{\alpha_i \alpha_j}{(\alpha_i + \alpha_j)^2} \cdot c_t \cdot \frac{\alpha_i + \alpha_j + c_t}{\alpha_i + \alpha_j + 1} . \tag{3.22}$$

What we introduced as γ must, therefore, be

$$\gamma_{ij} = \alpha_i + \alpha_j . \tag{3.23}$$

If inserting this in (3.17), we have

$$p\{a_{it}\} = (-1)^{a_{it}} \binom{-\alpha_i}{a_{it}} \cdot \left(\frac{\mu_{ijt}}{\alpha_i + \alpha_j + \mu_{ijt}}\right)^{a_{it}} \left(\frac{\alpha_i + \alpha_j}{\alpha_i + \alpha_j + \mu_{ijt}}\right)^{\alpha_i} , \tag{3.24}$$

but for general application of this formula, i.e. for all combinations of j and i, the ratio

$$\frac{\mu_{ijt}}{\alpha_i + \alpha_j} \tag{3.25}$$

must be independent of α_j, and by inversion of i and j it may also be seen that it must be independent of α_i, thus being dependent on t only. We can, therefore, say that

$$\mu_{ijt} = (\alpha_i + \alpha_j)\eta_t . \tag{3.26}$$

This provides us with the final formulation

$$(3.27) \qquad p\{a_{it}|\alpha_i,n_t\} = (-1)^{a_{it}} \binom{-\alpha_i}{a_{it}} \left(\frac{n_t}{1+n_t}\right)^{a_{it}} (1+n_t)^{-\alpha_i}$$

and also with those analogous for a_{jt} and c_t.

Pending the programming of a maximum likelihood estimation of the α's - the n's being of secondary importance in this connection - a more elementary method may be adopted.

From (3.21) describing the mean it follows that

$$(3.28) \qquad a_{io} = \sum_{(t)} a_{it}$$

has the mean value

$$(3.29) \qquad \mathscr{E}\{\sum_{(t)} a_{it}|(c_t)\} = \frac{\alpha_i}{\alpha_i+\alpha_j} \cdot c_o$$

so that

$$(3.30) \qquad \frac{\Sigma a_{it}}{\Sigma c_t} = \frac{a_{io}}{c_o} \approx \frac{\alpha_i}{\alpha_i+\alpha_j} \quad (=\theta_i)$$

and thus

$$(3.31) \qquad \frac{\Sigma a_{it}}{\Sigma a_{jt}} \approx \frac{\alpha_i}{\alpha_j}$$

Now (3.30) is an estimate for minimization of

$$(3.32) \qquad \Sigma \frac{(a_{it}-\theta_i c_t)^2}{c_t} = \Sigma \frac{(a_{it}-\hat{\theta}_i c_t)^2}{c_t} + (\hat{\theta}_i-\theta_i)^2 c_o ,$$

and we, therefore, seek the mean of this minimum. First, we have

$$(3.33) \qquad \mathscr{E}\{\Sigma \frac{(a_{it}-\theta_i c_t)^2}{c_t}|(c_t)\} = \Sigma \frac{1}{c_t} \mathscr{V}\{a_{it}|c_t\}$$

$$= \Sigma \frac{1}{c_t} \cdot c_t\theta_i(1-\theta_i) \frac{\gamma_{ij}+c_t}{\gamma_{ij}+1}$$

$$= \theta_i(1-\theta_i) \cdot (T + \frac{c_o-T}{\gamma_{ij}+1} ,$$

T being the number of specific days in the week during the period under observation. Then

$$c_o \cdot \mathcal{V}\{(\hat\theta_i-\theta_i)^2|(c_t)\} = c_o \, \mathcal{V}\{\hat\theta_i|(c_t)\}$$

$$= c_o \cdot \mathcal{V}\{\frac{\Sigma a_{it}}{c_o}|(c_t)\}$$

$$(3.34) \qquad = c_o \cdot \Sigma \frac{1}{c_o^2} \cdot \mathcal{V}\{a_{it}|c_t\}$$

$$= \frac{1}{c_o} \cdot \Sigma \; \theta_i(1-\theta_i) \cdot \frac{\gamma_{ij}^{+1} + c_t^{-1}}{\gamma_{ij}^{+1}} \cdot c_t$$

$$= \theta_i(1-\theta_i) \cdot (1+ \frac{1}{\gamma_{ij}+1} \cdot \frac{\Sigma c_t(c_t-1)}{\Sigma c_t})$$

and finally

$$(3.35) \quad \mathcal{M}\{\Sigma \frac{(a_{it}-\hat\theta_i c_t)^2}{c_t}|(c_t)\} = \theta(1-\theta_i) \cdot (T-1+ \frac{c_o-T-\frac{1}{c_o}\Sigma c_t(c_t-1)}{\gamma_{ij}+1}) ,$$

which leads directly to an estimation of $\frac{1}{\gamma_{ij}+1}$ with a view to arriving at the value which in the average number of cases would apply. From this an estimate is then derived as to the value of γ_{ij}, i.e. the absolute value of $\alpha_i + \alpha_j$. If multiplying the estimated values of

$$\theta_i = \frac{\alpha_i}{\alpha_i+\alpha_j} \text{ and } \theta_j = \frac{\alpha_j}{\alpha_i+\alpha_j} \text{ by } \hat\gamma_{ij} ,$$

we get an estimate of the absolute values of α_i and α_j.

The preciseness of these estimates may be evaluated, but this would require further algebraic developments which at the present time are impracticable.

§ 4. The Basis for Comparisons between Short Road Sections

The documentation set forth in (4, B § 3) is being advanced in an effort to demonstrate that the multiplicative Poisson law may be applied to the correlation of time and major sections of the road network.

In the Swedish investigation in 1965 (6) on the effect of increased traffic control on a couple of selected road segments, without introduction of speed limits, the evaluation of the results achieved is based on a comparison of simultaneous data from these segments which represent small sections of the Swedish "E" roads.

The question as to whether the number of accidents was reduced as a result of the increased traffic control will then depend on the distribution of accidents when comparing the road segments with and without increased control, respectively: Was a change experienced in the distribution in consequence of the increased control measures?

This problem invites a traditional u or χ^2-test for identification of the distribution of accidents on the road sections when comparing the sections with and without increased control for the two periods under review.

The fact remains, however, that these two periods comprise days marked by wide differences in traffic conditions, and moreover, it is a question of different seasons of the year so that, considered as a whole, the results must be expected to vary considerably.

In order to get a realistic picture when comparing the two types of road segments, the relative distribution of accidents on these two types must thus be stable for each period. According to the Chatterji converse theorem this would, however, mean that the multiplicative Poisson law would have to hold for both periods.

I have been considering whether it is possible, on basis of the validity of the theorem when applied to large road segments, to draw the conclusion that it must hold also for the smaller segments, but to the best of my knowledge there is no tenable mathematic basis for such an argument. A final opinion on the published material must, therefore, be postponed pending the availability of material stating the number of accidents per day on the respective two road sections.

In the absence of any such information, the evaluation must necessarily be subject to the following condition: <u>If the multiplicative Poisson law holds also for smaller sections of the road network, a traditional test will provide constructive results.</u>

We are, however, not completely without possibilities for control. Information as to the number of accidents on three other road sections during the two periods under review has been obtained from the official statistics, and also data on the investigated section for the previous year. If the multiplicative Poisson law holds for the individual days, it may also be concluded that it will hold for a comparison between the two periods of time, the only exception being the investigated road segment during the time of the investigation.

As the theorem seems to hold well (cf. (4), B § 6), the basic argument must also be assumed to hold, but it would still seem desirable to obtain data per day for the respective two road sectors as well as total figures for the 2 × 2 months for a considerable number of additional road segments.

List of References:

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