Please return to Ben Wright before Sept 13

August 1967

G.Rasch,Jon Stene

Some remarks concerning ~~the~~ inference about items with more than two categories.

ie. Pair wise estimation and fit!

1. The model.

Consider an item with m categories of response~~s~~, denoted by

$$(1.1) \qquad X : (x^{(1)},\ldots,x^{(\mu)},\ldots,x^{(m)}).$$

This item is given to an individual denoted by ν. A model for The probability of the response $x^{(\mu)}$ in this situation is ~~given by~~ ?

$$(1.2) \qquad p\{x^{(\mu)}|\nu,i\} = \frac{\xi_{\nu\mu}\varepsilon_{i\mu}}{\gamma_{\nu i}}, \quad \mu = 1,2,\ldots,m$$

where

$$(1.3) \qquad \gamma_{\nu i} = \sum_{\mu=1}^{m} \xi_{\nu\mu}\varepsilon_{i\mu}.$$

The vector

$$(1.4) \qquad \xi_\nu = (\xi_{\nu 1},\ldots,\xi_{\nu m}) \quad \xi_{\nu\mu}$$

is a parameter characterizing ~~the~~ individual ~~no.~~ ν and the vector

$$(1.5) \qquad \varepsilon_i = (\varepsilon_{i1},\ldots,\varepsilon_{im}) \quad \varepsilon_{i\mu}$$

characterizes item ~~no.~~ i.

By introducing the selection vector

$$(1.6) \qquad a_{\nu i} = (0,\ldots,0,\overset{\mu}{1},0,\ldots,0)$$

i.e. a vector of order m with a 1 as its μ'th component and zeros elsewhere, (1.2) may be written in the form

$$(1.7) \qquad p\{x^{(\mu)}|\nu,i\} = \frac{\xi_\nu^{a_{\nu i}}\varepsilon_i^{a_{\nu i}}}{\gamma_{\nu i}}$$

where

$$(1.8)\qquad \xi_\nu^{\,a_{\nu i}} = \xi_{\nu\mu}$$

and

$$(1.9)\qquad \varepsilon_i^{\,a_{\nu i}} = \varepsilon_{i\mu}$$

~~and~~ and ~~where~~ ξ_ν, ε_i and $a_{\nu i}$ are as given in (1.4), (1.5) and (1.6), respectively.

~~Let~~ a questionnaire with k items, each with m response categories ~~be~~ is given to N persons.

In this note some of the problems concerning ~~the~~ statistical inferences about the ε's of the different items will be discussed. When

2. On the estimation of the components of ~~the~~ item parameters

Let us consider two items i and j from the questionnaire. In the following table the observations are summarized. The different elements of the table denote the number of individuals with the corresponding combination of responses i.e. b_{gh} is the number of persons among the N observed who ~~have~~ gave ~~ones which has the~~ response g ~~to~~ item i and the response h ~~to~~ item j and b_{hg} is the number of persons ~~which has~~ who have gave the response h ~~to~~ item i and g ~~to~~ item j.

Item j

(2.1) Item i	$x^{(1)}$ …	$x^{(g)}$ …	$x^{(h)}$ …	$x^{(m)}$	
$x^{(1)}$	b_{11} …	b_{1g} …	b_{1h} …	b_{1m}	b_{1o}
	– –	– –	– –	– –	–
$x^{(g)}$	b_{g1} …	b_{gg} …	b_{gh} …	b_{gm}	b_{go}
	– –	– –	– –	– –	–
$x^{(h)}$	b_{h1} …	b_{hg} …	b_{hh} …	b_{hm}	b_{ho}
	– –	– –	– –	– –	–
$x^{(m)}$	b_{m1} …	b_{mg} …	b_{mh} …	b_{mm}	b_{mo}
	b_{o1} …	b_{og} …	b_{oh} …	b_{om}	

reader might
wonder why not
$2\prod_{g\neq h}$ (etc) as that would
— how could this
or $\prod_{g\neq h}$

cover the elements
be clarified —
b_{gh} actually always goes with b_{hg}
So there are only $\binom{m}{2}$ situations — for each
of which we have the prob. $p\{b_{gh} | n_{gh}\}$
— every person (falls / is counted) in one of these situations

So $\prod_{g<h}\left(\quad \right)$

~~Remark on this definition of marginals - what does it imply. Why is it done.~~

where the marginals are defined as

$$(2.2) \qquad b_{go} = \sum_{h=1}^{m} b_{gh} - b_{gg} = \sum_{h \neq g} b_{gh}$$

~~and~~ and

$$(2.3) \qquad b_{oh} = \sum_{g=1}^{m} b_{gh} - b_{hh} = \sum_{g \neq h} b_{gh} \, ,$$

and

$$(2.4) \qquad N = \sum_{g=1}^{m} \sum_{h=1}^{m} b_{gh} \; .$$

The selection vector (1.6) will ~~in the sequel~~ be denoted by e_g when the response is at the category g, and e_h when it is at category h, i.e.

$$(2.5) \qquad a_{\nu i} = (o,\ldots,o,\overset{g}{1},o,\ldots,o,\overset{h}{o},o,\ldots,o) = e_g$$

$$(2.6) \qquad a_{\nu i} = (o,\ldots,o,\overset{g}{o},o,\ldots,o,\overset{h}{1},o,\ldots,o) = e_h$$

why not show derivation

Insert derivation

From (1.7) we derive by means of (2.5) and (2.6) that the ~~conditional~~ probability of ~~the~~ response g ~~to~~ item i and h ~~to~~ item j, given that ~~the~~ individual ~~no.~~ ν has ~~given~~ made exactly one g-response and one h-response on these two items.

$$p\{a_{\nu i} = e_g, a_{\nu j} = e_h \,|\, a_{\nu i} + a_{\nu j} = e_g + e_h\} = \frac{\varepsilon_i^{e_g} \, \varepsilon_j^{e_h}}{\varepsilon_i^{e_g}\varepsilon_j^{e_h} + \varepsilon_i^{e_h}\varepsilon_j^{e_g}}$$

$$(2.7) \qquad = \frac{\varepsilon_{ig}\,\varepsilon_{jh}}{\varepsilon_{ig}\,\varepsilon_{jh} + \varepsilon_{ih}\,\varepsilon_{jg}} = \frac{\dfrac{\varepsilon_{ig}}{\varepsilon_{jg}}}{\dfrac{\varepsilon_{ig}}{\varepsilon_{jg}} + \dfrac{\varepsilon_{ih}}{\varepsilon_{jh}}} \; .$$

By introducing

$$(2.8) \qquad \frac{\varepsilon_{ig}}{\varepsilon_{jg}} = \delta_g \quad \text{and} \quad \frac{\varepsilon_{ih}}{\varepsilon_{jh}} = \delta_h$$

(2.7) takes the form

$$(2.9) \qquad p\left\{a_{\nu i} = e_g,\ a_{\nu j} = e_h \mid a_{\nu i} + a_{\nu j} = e_g + e_h\right\} = \frac{\delta_g}{\delta_g + \delta_h}$$

Let $g \neq h$, (otherwise (2.7) is equal to $\frac{1}{2}$ and there is no information contained), and let

explain implications

$$(2.10) \qquad b_{gh} + b_{hg} = n_{gh}$$

From (2.9) (+) follows by the binomial law, that

$$(2.11) \qquad p\left\{b_{gh}, b_{hg} \mid n_{gh}\right\} = \binom{n_{gh}}{b_{gh}} \frac{\delta_g^{\,b_{gh}}\ \delta_h^{\,b_{hg}}}{(\delta_g + \delta_h)^{n_{gh}}}$$

stet

Since all ~~the~~ N persons are considered to react ~~stochastically~~ independently and each person is contained in ~~exactly~~ Some one of the elements of (2.1), it follows that the probability of the actual set of results outside the diagonal in (2.1) is

$$(2.12) \qquad p\left\{((b_{gh}, b_{hg})) \mid ((n_{gh}))\right\} = \prod_{g<h} \binom{n_{gh}}{b_{gh}} \frac{\prod\limits_{g<h} \delta_g^{\,b_{gh}}\ \prod\limits_{g<h} \delta_h^{\,b_{hg} = n_{gh} - b_{gh}}}{\prod\limits_{g<h} (\delta_g + \delta_h)^{n_{gh}}}$$

where $((b_{gh}, b_{hg}))$ is the whole set of corresponding pairs laying symmetrically about the diagonal and $((n_{gh}))$ is the set of n_{gh}'s.

It is seen that

$$(2.13) \qquad \sum_{h=2}^{m} \sum_{g<h} n_{gh} = N - \sum_{g=1}^{m} b_{gg} = \sum_{g=1}^{m} b_{go} = \sum_{h=1}^{m} b_{oh}$$

So what — can you make something of it

new page

In order to make the idea simple, consider the case m=3. Here (2.1) takes the form

(2.14)

		j: $x^{(1)}$	$x^{(2)}$	$x^{(3)}$	
i	$x^{(1)}$	b_{11}	b_{12}	b_{13}	$b_{10} = b_{12} + b_{13}$
	$x^{(2)}$	b_{21}	b_{22}	b_{23}	b_{20} etc.
	$x^{(3)}$	b_{31}	b_{32}	b_{33}	b_{30}
		b_{01} $= b_{21} + b_{31}$	b_{02} etc.	b_{03}	N

The marginals are defined in (2.2),(2.3) and (2.4). The probability (2.12) is then ~~reduced to the form~~

$$p\{(b_{12},b_{21}),(b_{13},b_{31}),(b_{23},b_{32})\,|\,n_{12},n_{13},n_{23}\} =$$

$$=\binom{n_{12}}{b_{12}}\binom{n_{13}}{b_{13}}\binom{n_{23}}{b_{23}}\frac{\delta_1^{b_{12}}\,\delta_2^{b_{21}}}{(\delta_1+\delta_2)^{n_{12}}}\frac{\delta_1^{b_{13}}\,\delta_3^{b_{31}}}{(\delta_1+\delta_3)^{n_{13}}}\frac{\delta_2^{b_{23}}\,\delta_3^{b_{32}}}{(\delta_2+\delta_3)^{n_{23}}}$$

collecting $\delta_1, \delta_2, \delta_3$

(2.15)

$$=\binom{n_{12}}{b_{12}}\binom{n_{13}}{b_{13}}\binom{n_{23}}{b_{23}}\frac{\delta_1^{b_{12}+b_{13}}\,\delta_2^{b_{21}+b_{23}}\,\delta_3^{b_{31}+b_{32}}}{(\delta_1+\delta_2)^{n_{12}}(\delta_1+\delta_3)^{n_{13}}(\delta_2+\delta_3)^{n_{23}}}$$

$$=\binom{n_{12}}{b_{12}}\binom{n_{13}}{b_{13}}\binom{n_{23}}{b_{23}}\frac{\delta_1^{b_{1o}}\,\delta_2^{b_{2o}}\,\delta_3^{b_{3o}}}{(\delta_1+\delta_2)^{n_{12}}(\delta_1+\delta_3)^{n_{13}}(\delta_2+\delta_3)^{n_{23}}}$$

This expression is homogenous in the δ's therefore only the relations between them may be estimated. In order to carry out the estimation some restriction has to be ~~done,~~ specified e.g.

(2.16) $$\delta_1\ \delta_2\ \delta_3 \Rightarrow 1$$

or

(2.17) $$\delta_3 \Rightarrow 1 .$$

why not say a word about relative merits of these (— or indifference)

If $\delta_3 \Rightarrow 1$ ~~In the last case~~ the right side of (2.15) can be written in the form

(2.18) $$\binom{n_{12}}{b_{12}}\binom{n_{13}}{b_{13}}\binom{n_{23}}{b_{23}}\frac{\delta_1'^{\,b_{1o}}\ \delta_2'^{\,b_{2o}}}{(\delta_1'+\delta_2')^{n_{12}}\,(\delta_1'+1)^{n_{13}}(\delta_2'+1)^{n_{23}}}$$

from which the following set of maximum likelihood equations for estimating

(2.19) $$\delta_1' = \frac{\delta_1}{\delta_3} = \frac{\varepsilon_{i1}}{\varepsilon_{j1}} \Big/ \frac{\varepsilon_{i3}}{\varepsilon_{j3}}$$

and

explain

Then combine

$$\left.\begin{aligned} \frac{b_{12}}{n_{12}} &\approx \frac{\delta_1}{\delta_1+\delta_2} \\ \frac{b_{13}}{n_{13}} &\approx \frac{\delta_1}{\delta_1+\delta_3} \end{aligned}\right\} \quad \frac{b_{12}+b_{13}}{\delta_1} \approx \frac{n_{12}}{\delta_1+\delta_2} + \frac{n_{13}}{\delta_1+\delta_3}$$

$b_{12}\delta_1 + b_{12}\delta_2 \approx n_{12}\delta_1 \qquad b_{21}\delta_2 + b_{21}\delta_1 \approx n_{12}\delta_2$

$b_{13}\delta_1 + b_{13}\delta_3 \approx n_{13}\delta_1 \qquad b_{23}\delta_2 + b_{23}\delta_3 \approx n_{23}\delta_2$

$\delta_2 \approx \delta_1(n_{12}-b_{12})/b_{12} = \delta_1(b_{21}/b_{12})$

$\delta_3 \approx \delta_1(b_{31}/b_{13})$

$\delta_3 \approx \delta_2(b_{32}/b_{23})$

or $\boxed{\begin{aligned} \delta_1/\delta_2 &\approx b_{12}/b_{21} \\ \delta_1/\delta_3 &\approx b_{13}/b_{31} \\ \delta_2/\delta_3 &\approx b_{23}/b_{32} \end{aligned}}$

$$\begin{aligned} \log\delta_1 - \log\delta_1 &\approx \log(b_{11}/b_{11}) = l_{11} = 0 \\ \log\delta_1 - \log\delta_2 &\approx \log(b_{12}/b_{21}) = l_{12} \\ \log\delta_1 - \log\delta_3 &\approx \qquad = l_{13} \\ \log\delta_2 - \log\delta_3 &\approx \qquad = l_{23} \end{aligned}$$

note $3\log\delta_1 - \underbrace{(\log\delta_1 + \log\delta_2 + \log\delta_3)}_{\Rightarrow 0} \approx l_{11} + l_{12} + l_{13} = l_{1+}$

so $\log\delta_1 \approx l_{1\cdot}$ or in general $\log\delta_g \approx l_{g\cdot}$

$\delta_g = \epsilon_{ig}/\epsilon_{jg}$

$\log\delta_g = \log\epsilon_{ig} - \log\epsilon_{jg} \approx l_{ijg\cdot}$

$l_{ijgh} = \log(b_{ijgh}/b_{ijhg})$

$l_{ijg\cdot} = \sum_h^m l_{ijgh}/m$

so if $\sum_g^m \log\delta_g \Rightarrow 0$

$\sum_g^L \sum_i^k \log\epsilon_{ig} \Rightarrow 0$

$\boxed{l_{i\cdot g\cdot} \approx \log\epsilon_{ig}}$ see program BIGPAR

Wouldn't it be nice to show the development of these — to differentiate etc. and also as on

(2.20) $$\delta'_2 = \frac{\delta_2}{\delta_3} = \frac{\varepsilon_{i2}}{\varepsilon_{j2}} \Big/ \frac{\varepsilon_{i3}}{\varepsilon_{j3}}$$

are derived

(2.21)
$$\frac{b_{1o}}{\delta'_1} \approx \frac{n_{12}}{\delta'_1 + \delta'_2} + \frac{n_{13}}{\delta'_1 + 1}$$
$$\frac{b_{2o}}{\delta'_2} \approx \frac{n_{12}}{\delta'_1 + \delta'_2} + \frac{n_{23}}{\delta'_2 + 1}$$

Returning to the general case ~~it is seen~~ that formula (2.12) may be written in the form

(2.22)
$$p\{((b_{gh}, b_{hg}))|((n_{gh}))\} = \prod_{g=1}^{m} \prod_{h>g} \binom{n_{gh}}{b_{gh}} \frac{\prod_{g=1}^{m} \prod_{h \neq g} \delta_g^{b_{gh}}}{\prod_{g=1}^{m} \prod_{h>g} (\delta_g + \delta_h)^{n_{gh}}}$$

ok $h \neq g$

$$= \prod_{g=1}^{m} \prod_{h>g} \binom{n_{gh}}{b_{gh}} \frac{\prod_{g=1}^{m} \delta_g^{b_{go}}}{\prod_{g=1}^{m} \prod_{h>g} (\delta_g + \delta_h)^{n_{gh}}}$$

since $b_{go} = \sum_{h \neq g}^{m} b_{gh}$

If we make the restriction

(2.23) $$\delta_m = 1$$

Wouldn't you like to show what happens with the other restriction? $\prod_{g}^{m} \delta_g = 1$

we may derive the following set of maximum likelihood equations to estimate

(2.24) $$\delta'_g = \frac{\delta_g}{\delta_m}, \quad g = 1,2,\ldots,m-1.$$

Show how

(2.25) $$\frac{b_{go}}{\delta'_g} \approx \sum_{\alpha=1}^{g-1} \frac{n_{\alpha g}}{(\delta'_\alpha + \delta'_g)} + \sum_{\beta=g+1}^{m-1} \frac{n_{g\beta}}{(\delta'_g + \delta'_\beta)} + \frac{n_{gm}}{(\delta'_g + 1)} \qquad g = 1,2,\ldots,m-1$$

$$\left(\sum_{h=1}^{m} b_{gh} - b_{gg}\right) \approx \delta_g\left(\sum_{h=1}^{m}\left(n_{gh}\Big/(\delta_g+\delta_h)\right)\right) - \delta_g n_{gg}\Big/(\delta_g+\delta_g)$$

Over

if $n_{gg} \Rightarrow b_{gg} + b_{gg} = 2 b_{gg}$

then $\sum_{h=1}^{ng} b_{gh} \approx \sum_{h=1}^{ng} \delta_g n_{gh} / (\delta_g + \delta_h)$

would $\delta_g \approx \sum_{h=1}^{ng} b_{gh} \Big/ \sum_{h=1}^{ng} \left(n_{gh} / (\delta_g + \delta_h) \right)$ iterated

be any use?

or solve this with Newtons method for δ_g

$F = \delta_g - b_{g+} \Big/ \sum_h \left(n_{gh} / (\delta_g + \delta_h) \right) = \delta_g - b_{g+} / \gamma_g$

$F' = 1 - (b_{g+})(-1)\gamma_g^{-2}(-1)(n_{gh})(\delta_g + \delta_h)^{-2}$ what about $g = h$?

so redefine γ_g $F = \delta_g - \sum_{h \neq g} b_{gh} \Big/ \sum_{h \neq g} \left(n_{gh} / (\delta_g + \delta_h) \right) = \delta_g - b_{g+} / \gamma_g$

$F' = 1 - \left(\sum_{h \neq g} b_{gh} \right)(-1)\left(\gamma_g^{-2} \right)\left(\sum_{h \neq g} n_{gh} (-1) (\delta_g + \delta_h)^{-2} \right)$?

which may be solved by usual numerical methods. Straw one

3. Control of the model.

Consider first the case m=3. Formula (2.15) shows that (n_{12},n_{13},n_{23}) and (b_{1o},b_{2o},b_{3o}) together form a set of sufficient estimators for the model. The probability for the obtained set (b_{1o},b_{2o},b_{3o}) when the set (n_{12},n_{13},n_{23}) is given is

$$p\{(b_{1o},b_{2o},b_{3o})|(n_{12},n_{13},n_{23})\} =$$

$$(3.1) \qquad \frac{\delta_1^{b_{1o}}\,\delta_2^{b_{2o}}\,\delta_3^{b_{3o}}}{(\delta_1+\delta_2)^{n_{12}}(\delta_1+\delta_3)^{n_{13}}(\delta_2+\delta_3)^{n_{23}}} \sum_{\substack{b_{12}+b_{13}=b_{1o}\\ b_{21}+b_{23}=b_{2o}\\ b_{31}+b_{32}=b_{3o}}} \binom{n_{12}}{b_{12}}\binom{n_{13}}{b_{13}}\binom{n_{23}}{b_{23}}$$

Here we introduce the notation

$$(3.2) \quad \gamma((b_{1o},b_{2o},b_{3o})|(n_{12},n_{13},n_{23})) = \sum_{\substack{b_{12}+b_{13}=b_{1o}\\ b_{21}+b_{23}=b_{2o}\\ b_{31}+b_{32}=b_{3o}}} \binom{n_{12}}{b_{12}}\binom{n_{13}}{b_{13}}\binom{n_{23}}{b_{23}}$$

The conditional probability of the obtained $((b_{gh},b_{hg}))$ given the sufficient estimators are then derived from (3.1), ~~and~~ (3.2) and (2.15)

$$(3.3) \quad p\{((b_{gh},b_{hg}))|(b_{go}),((n_{gh}))\} = \frac{\binom{n_{12}}{b_{12}}\binom{n_{13}}{b_{13}}\binom{n_{23}}{b_{23}}}{\gamma((b_{1o},b_{2o},b_{3o})|(n_{12},n_{13},n_{23}))}$$

where the δ's are eliminated. By means of (3.3) a nonparametric control of the model may take place. One

As an example of such a control consider the following. ~~example~~. Let the items i and j be given to two different groups of individuals. The ~~hypothesis~~ to be tested is that the relation between the two items is the same for the two groups, i.e.

$$(3.4) \quad \delta_{11} = \delta_{12} = \delta_1, \quad \delta_{21} = \delta_{22} = \delta_2, \; \delta_{31} = \delta_{32} = \delta_3 .$$

assumption of the model

control depends on <u>criteria</u>

<u>systematic partition of data</u>

to challenge sample-freedom

a) divide persons } statistical test identity of estimates

b) divide items

~~which is one of the assumptions of the model~~. Let the elements of (2.14) be denoted by b'_{gh} for the first group and ~~the corresponding elements~~ for the second ~~group~~ by b''_{gh}.

For the first group we get

$$(3.5)\quad p\{((b'_{gh}, b'_{hg}))\,|\,((n'_{gh}))\}$$

$$= \binom{n'_{12}}{b'_{12}}\binom{n'_{13}}{b'_{13}}\binom{n'_{23}}{b'_{23}} \frac{\delta_{11}^{b_{10}}\ \delta_{21}^{b_{20}}\ \delta_{31}^{b_{30}}}{(\delta_{11}+\delta_{21})^{n'_{12}}\ (\delta_{11}+\delta_{31})^{n'_{13}}\ (\delta_{21}+\delta_{31})^{n'_{23}}}$$

and hence

$$p\{(b'_{10}, b'_{20}, b'_{30})\,|\,(n'_{12}, n'_{13}, n'_{23})\} =$$

$$(3.6)\quad = \frac{\delta_{11}^{b'_{10}}\ \delta_{21}^{b'_{20}}\ \delta_{31}^{b'_{30}}}{(\delta_{11}+\delta_{21})^{n'_{12}}\ (\delta_{11}+\delta_{31})^{n'_{13}}\ (\delta_{21}+\delta_{31})^{n'_{23}}} \cdot \gamma((b'_{go})\,|\,((n'_{gh})))$$

For the second group the corresponding probability is

$$p\{(b''_{10}, b''_{20}, b''_{30})\,|\,(n''_{12}, n''_{13}, n''_{23})\} =$$

$$(3.7)\quad = \frac{\delta_{12}^{b''_{10}}\ \delta_{22}^{b''_{20}}\ \delta_{32}^{b''_{30}}}{(\delta_{12}+\delta_{22})^{n''_{12}}\ (\delta_{12}+\delta_{32})^{n''_{13}}\ (\delta_{22}+\delta_{32})^{n''_{23}}} \cdot \gamma((b''_{go})\,|\,((n''_{gh})))$$

If the ~~hypothesis~~ assumption (3.4) holds, then ~~the~~ this probability is also given by (3.1). The conditional probability for the obtained results in the two groups given the total result ~~is since~~ is

where

$$(3.8)\quad b'_{gh} + b''_{gh} = b_{gh},\quad b'_{go} + b''_{go} = b_{go},\quad n'_{gh}+n''_{gh}=n_{gh} \text{ for all } (g,h);$$

which follows
~~easily derived~~ from (3.1), (3.2), (3.5) and (3.6) 15

$$p\{(b'_{go}),((n'_{gh})),(b''_{go}),((n''_{gh}))|(b_{go}),((n_{gh}))\} =$$

(3.9)
$$= \frac{\gamma((b'_{go})|((n'_{gh}))) \cdot \gamma((b''_{go})|((n''_{gh})))}{\gamma((b_{go})|((n_{gh})))}$$

where the δ's are eliminated. If (3.9) is too small the hypothesis (3.4) is rejected.

What about the set of smaller (3.9)'s

It is possible to carry out this procedure for all pairs of items.

If the probability (3.9) is small for a large number of these pairs, the conclusion is that the two groups react different to the items, and the model assumptions are not fulfilled.

Consider then the general case.
From (2.22) it is derived that

$$p\{(b_{go})|((n_{gh}))\} = \frac{\prod_{g=1}^{m} \delta_g^{b_{go}}}{\prod_{g=1}^{m} \prod_{h>g} (\delta_g+\delta_h)^{n_{gh}}} \cdot \gamma((b_{go})|((n_{gh}))) \tag{3.10}$$

where

$$\gamma((b_{go})|((n_{gh}))) = \sum_{(\sum_{h=1, h\neq g}^{m} b_{gh}=b_{go};\, g=1,\dots,m)} \prod_{g=1}^{m} \prod_{h>g} \binom{n_{gh}}{b_{gh}} \tag{3.11}$$

If the two items are presented to two different groups, it may be of interest to test whether the itemparameters are equal for the two groups or not, i.e.

$$\delta_{11} = \delta_{12} = \delta_1, \dots, \quad \delta_{m1} = \delta_{m2} = \delta_m \tag{3.12}$$

The testing procedure is carried through in the same way as for m=3, i.e. we consider

$$p\left\{(b'_{go}),((n'_{gh})),(b''_{go}),((n''_{gh}))|(b_{go}),((n_{gh}))\right\} =$$

(3.13)

$$= \frac{\gamma((b'_{go})|((n'_{gh}))) \cdot \gamma((b''_{go})|((n''_{gh})))}{\gamma((b_{go})|((n_{gh})))}$$

where the different γ's in (3.13) are formed in analogy to (3.11).

This method may easily be generalized to other types of control. for example ?

1) for the sake of education — wherever you say "may desire" - it would be better to show how to do it

2) why not include ~~the~~ a method of estimation and the approx. std. errors of these estimates