Nonparametric Bounds Analysis for NEAT Equating

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Outline

1. The equating statistical problem
2. NEAT equating design: the current and an alternative view
3. Illustration
4. Discussion
Test forms $X$ and $Y$ are administrated to $n_x$ and $n_y$ test takers, respectively.

The scores $X$ and $Y$ are assumed to be random variables.

$X$ and $Y$ are defined on (score) sample spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively.

$x_1, \ldots, x_{n_x} \sim F_X(x)$ and $y_1, \ldots, y_{n_y} \sim F_Y(y)$. 

Statistical problem

Modeling the relationship between scores to make them comparable.
General formulation of the equating problem

- Test forms X and Y are administered to \( n_x \) and \( n_y \) test takers, respectively.
- The scores \( X \) and \( Y \) are assumed to be random variables.
- \( X \) and \( Y \) are defined on (score) sample spaces \( X \) and \( Y \), respectively.
- \( x_1, \ldots, x_{n_x} \sim F_X(x) \) and \( y_1, \ldots, y_{n_y} \sim F_Y(y) \).

Statistical problem

Modeling the relationship between scores to make them comparable.
General formulation: equating transformation

Definition

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sample spaces. A function $\varphi : \mathcal{X} \mapsto \mathcal{Y}$ will be called an equating transformation.

- The equating transformation maps the scores on the scale of one test form into the scale of the other.
- The equating transformation is to be estimated by an estimator $\varphi_n$ based on samples $x_1, \ldots, x_{n_x} \sim F_X$ and $y_1, \ldots, y_{n_y} \sim F_Y$, respectively.
Equipercentile transformation

\[ y = \varphi(x) = F_Y^{-1}(F_X(x)) \]
Equipercentile transformation

\[
y = \varphi(x) = F_Y^{-1}(F_X(x))
\]
NEAT equating design

<table>
<thead>
<tr>
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<tr>
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</tr>
<tr>
<td>Q</td>
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\[ T = w_P P + w_Q Q \]

\[ f_{XT}(x) = w_P f_{XP}(x) + w_Q f_{XQ}(x) \]

\[ f_{YT}(y) = w_P f_{YP}(y) + w_Q f_{YQ}(y). \]

Additional assumptions are needed to estimate \( f_{XT}(x) \) and \( f_{YT}(y) \).
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- Additional assumptions are needed to estimate \(f_{XT}(x)\) and \(f_{YT}(y)\)
Most common assumption,

\[ f_{XP}(x \mid a) = f_{XQ}(x \mid a) \quad \text{and} \quad f_{YP}(y \mid a) = f_{YQ}(y \mid a) \]

Using these assumptions we get

\[ f_{XT}(x) = w_P f_{XP}(x) + w_Q \sum_a f_{XP}(x) f_{AQ}(a) \]
\[ f_{YT}(y) = w_P \sum_a f_{YQ}(y) f_{AP}(a) + w_Q f_{YQ}(y) \]

and from here

\[ \varphi_T(x) = F_{YT}^{-1}(FX_T(x)) \]
Most common assumption,

\( f_{XP}(x \mid a) = f_{XQ}(x \mid a) \) and \( f_{YP}(y \mid a) = f_{YQ}(y \mid a) \)

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and from here

\[
\varphi_T(x) = F_{YT}^{-1}(F_{XT}(x))
\]
Conditional score distributions with no assumptions

- Let $Z$ denote the population group so that

\[
Z = \begin{cases} 
1, & \text{if test taker is administered } X; \\
0, & \text{if test taker is administered } Y.
\end{cases}
\]

- Then, by the Law of Total Probability (LTP)

\[
P(X \leq x \mid A) = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A) + P(X \leq x \mid A, Z = 0)P(Z = 0 \mid A)
\]

- $P(X \leq x \mid A)$ is not identified.
Conditional score distributions with no assumptions

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$$P(X \leq x \mid A) = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A) + P(X \leq x \mid A, Z = 0)P(Z = 0 \mid A) \quad (1)$$

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Let $Z$ denote the population group so that

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1, & \text{if test taker is administered X;} \\
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$P(X \leq x \mid A)$ is not identified.
Partially identified probability distributions

- From (1) and the fact that $P(X \leq x \mid A, Z = 0)$ is bounded between 0 and 1, it follows that

\[ L_x \leq P(X \leq x \mid A) \leq U_x \]  

(2)

\[ L_x = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A) \]
\[ U_x = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A) + P(Z = 0 \mid A) \]

- Analogously for $Y$ we have

\[ L_y \leq P(Y \leq y \mid A) \leq U_y \]  

(3)

\[ L_y = P(Y \leq y \mid A, Z = 0)P(Z = 0 \mid A) \]
\[ U_y = P(Y \leq y \mid A, Z = 0)P(Z = 0 \mid A) + P(Z = 1 \mid A) \]
From (1) and the fact that $P(X \leq x \mid A, Z = 0)$ is bounded between 0 and 1, it follows that

$$L_x \leq P(X \leq x \mid A) \leq U_x \tag{2}$$

$L_x = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A)$

$U_x = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A) + P(Z = 0 \mid A)$

Analogously for $Y$ we have

$$L_y \leq P(Y \leq y \mid A) \leq U_y \tag{3}$$

$L_y = P(Y \leq y \mid A, Z = 0)P(Z = 0 \mid A)$

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(3)

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$U_y = P(Y \leq y \mid A, Z = 0)P(Z = 0 \mid A) + P(Z = 1 \mid A)$
Graphical illustration

Bounds for $P(X < x | A = 2)$ (black line)
Bounds for $P(Y < y | A = 2)$ (red line)
Target distributions with no assumptions

- Marginalizing over $A$,

\[
P(X \leq x \mid A) = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A) + P(X \leq x \mid A, Z = 0)P(Z = 0 \mid A)
\]

becomes

\[
P(X \leq x) = P(X \leq x \mid Z = 1)P(Z = 1) + P(X \leq x \mid Z = 0)P(Z = 0)
\]

or

\[
F_X(x) = w_P F_{XP}(x) + w_Q F_{XQ}(x)
\]

- Problem: $F_{XQ}(x) = P(X \leq x \mid Z = 0)$ is still non identified!
Target distributions with no assumptions

- Marginalizing over $A$,

\[
P(X \leq x \mid A) = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A) + \]
\[
P(X \leq x \mid A, Z = 0)P(Z = 0 \mid A)
\]

becomes

\[
P(X \leq x) = P(X \leq x \mid Z = 1)P(Z = 1) + P(X \leq x \mid Z = 0)P(Z = 0)
\]

or

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F_X(x) = w_P F_{XP}(x) + w_Q F_{XQ}(x)
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\[ F_X(x) = w_P F_{XP}(x) + w_Q F_{XQ}(x) \]

- Problem: $F_{XQ}(x) = P(X \leq x \mid Z = 0)$ is still non identified!
Partially identified target distributions

- $P(X \leq x)$ can also be bounded. In fact,

$$L_x \leq P(X \leq x) \leq U_x,$$

$$L_x = P(X \leq x \mid Z = 1)P(Z = 1)$$

$$U_x = P(X \leq x \mid Z = 1)P(Z = 1) + P(Z = 0)$$

- How does $F_X(x)$ compare to $F_{XT}(x)$?
Partially identified target distributions

- $P(X \leq x)$ can also be bounded. In fact,

$$L_x \leq P(X \leq x) \leq U_x,$$

$$L_x = P(X \leq x \mid Z = 1)P(Z = 1)$$
$$U_x = P(X \leq x \mid Z = 1)P(Z = 1) + P(Z = 0)$$

- How does $F_X(x)$ compare to $F_{XT}(x)$?
### Bounds illustrations

<table>
<thead>
<tr>
<th>Score</th>
<th>$F_{XT}$</th>
<th>$[L_x, U_x]$</th>
<th>$F_{YT}$</th>
<th>$[L_y, U_y]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.100</td>
<td>[0.050 ; 0.550]</td>
<td>0.105</td>
<td>[0.040 ; 0.540]</td>
</tr>
<tr>
<td>1</td>
<td>0.250</td>
<td>[0.125 ; 0.625]</td>
<td>0.320</td>
<td>[0.140 ; 0.640]</td>
</tr>
<tr>
<td>2</td>
<td>0.500</td>
<td>[0.250 ; 0.750]</td>
<td>0.530</td>
<td>[0.250 ; 0.750]</td>
</tr>
<tr>
<td>3</td>
<td>0.750</td>
<td>[0.375 ; 0.875]</td>
<td>0.755</td>
<td>[0.375 ; 0.875]</td>
</tr>
<tr>
<td>4</td>
<td>0.900</td>
<td>[0.450 ; 0.950]</td>
<td>0.900</td>
<td>[0.450 ; 0.950]</td>
</tr>
<tr>
<td>5</td>
<td>1.000</td>
<td>[0.500 ; 1.000]</td>
<td>1.000</td>
<td>[0.500 ; 1.000]</td>
</tr>
</tbody>
</table>
Graphical illustration

- Bounds for $F(x)$
- Bounds for $F(y)$
- $F_T(x)$
- $F_T(y)$

Scores

Pr

0 1 2 3 4 5

0.0 0.2 0.4 0.6 0.8 1.0

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Bounded quantiles

Let $\alpha \in (0, 1)$ and $q_\alpha(X) \doteq \inf\{t : P(X \leq t) > \alpha\}$. Define the following quantiles:

$$r_\alpha(X) \doteq \inf\{t : P(X \leq t \mid Z = 1)P(Z = 1) + P(Z = 0) > \alpha\}$$

$$= \inf \left\{ t : P(X \leq t \mid Z = 1) > \frac{\alpha - P(Z = 0)}{P(Z = 1)} \right\}$$

$$= q_\alpha^*(X \mid Z = 1)$$

and

$$s_\alpha(X) \doteq \inf\{t : P(X \leq t \mid Z = 1)P(Z = 1) > \alpha\}$$

$$= \inf \left\{ t : P(X \leq t \mid Z = 1) > \frac{\alpha}{P(Z = 1)} \right\}$$

$$= q_{\alpha'}(X \mid Z = 1)$$
In the NEAT design, the quantiles of the partially identified probabilities $P(X \leq t)$ and $P(Y \leq t)$ are also partially identified by the following intervals:

\begin{align*}
(i) & \quad r_{\alpha}(X) \leq q_{\alpha}(X) \leq s_{\alpha}(X); \\
(ii) & \quad r_{\alpha}(Y) \leq q_{\alpha}(Y) \leq s_{\alpha}(Y). 
\end{align*}

(5)
### Main idea

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( r_{\alpha}(X) ), ( s_{\alpha}(X) )</th>
<th>( r_{\alpha}(Y) ), ( s_{\alpha}(Y) )</th>
</tr>
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<tbody>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>0.1</td>
<td>[1 ; 3]</td>
<td>[2 ; 4]</td>
</tr>
<tr>
<td>0.2</td>
<td>[2 ; 4]</td>
<td>[3 ; 5]</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
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The lower and the upper bounds are not always informative:

\[
0 \leq \frac{\alpha - P(Z=0)}{P(Z=1)} \leq 1, \quad \text{for all } \alpha \in [P(Z = 0), 1];
\]

\[
0 \leq \frac{\alpha}{P(Z=1)} \leq 1, \quad \text{for all } \alpha \in [0, P(Z = 1)].
\]
Bounded equipercentile equating

- Given that

\[ F_{XP}(t) \cdot w \leq P(X \leq t) \leq F_{XP}(t) \cdot w + (1 - w) \]

then

\[ F_{Y}^{-1}(F_{XP}(t) \cdot w) \leq \varphi(t) \leq F_{Y}^{-1}(F_{XP}(t) \cdot w + (1 - w)) \]

moreover

\[ u_{Y}(\alpha) = \inf\{ t : F_{Y}(t) > F_{YQ}(t) \cdot (1 - w) > \alpha \} \]

and

\[ l_{Y}(\alpha) = \inf\{ t : F_{YQ}(t) \cdot (1 - w) + w > F_{Y}(t) > \alpha \} \]
Data from Kolen and Brennan (2014). Two 36-items test forms. Form X was administered to 1,655 examinees and form Y was administered to 1,638 examinees. Also, 12 out of the 36 items are common between both test forms (items 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, and 36).

\[ w_P = 1 \text{ and } w_Q = 0 \]
Graphical illustration

Score distributions in P and Q

F_XP
F_YQ

González & San Martín

NEAT equating
The equating statistical problem
NEAT equating design: the current and an alternative view
Illustration
Discussion
References

Graphical illustration

Score distributions in P and Q

- F_XP
- F_YQ
- F_XT (w1=1)
- F_YT (w2=0)

González & San Martín

NEAT equating
Graphical illustration

Score distributions in P and Q

- *F*<sub>XP</sub>
- *F*<sub>YQ</sub>
- *F*<sub>XT (w1=0.7)</sub>
- *F*<sub>YT (w2=0.3)</sub>
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Graphical illustration

Score distributions in P and Q

F_{XP}
F_{YQ}
F_{XT} (w1=0.5)
F_{YT} (w2=0.5)

0 5 10 15 20 25 30...
Graphical illustration

Score distributions in P and Q

- $F_{XP}$
- $F_{YQ}$
- $F_{XT} (w_1=0.3)$
- $F_{YT} (w_2=0.7)$

Scores
Graphical illustration

Score distributions in P and Q

- $F_{XP}$
- $F_{YQ}$
- $F_{XT}(w1=0)$
- $F_{YT}(w2=1)$
Graphical illustration

Score distributions

- $F_{XT}(w1=1)$
- $F_{YT}(w2=0)$
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Graphical illustration

Score distributions

- $F_{XT}$ (w1=1)
- $F_{YT}$ (w2=0)
- $U_x$
- $L_x$
- $U_y$
- $L_y$

Scores
Graphical illustration
Summary and discussion

- The equality of conditional distributions assumption is actually an *identification restriction*.
- We offer an alternative to the conditional distributions restriction → Partially identified probability distributions
- The derived bounds show that there is huge uncertainty about the probability distributions that are to be used for equating
Future work

- Equiquantile vs equipercentile: which one?
- Incorporate additional information in order to obtain tighter bounds.
Thank you for your attention!

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