Nonparametric Bounds Analysis for NEAT Equating

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Seminario de Estadística Educacional
Septiembre 1, 2017, PUC, Chile

Outline

General formulation of the equating problem

- Test forms X and Y are administrated to n_x and n_y test takers, respectively.
- The scores X and Y are assumed to be random variables.
- X and Y are defined on (score) sample spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively.
- $x_1, \ldots, x_{n_x} \sim F_X(x)$ and $y_1, \ldots, y_{n_y} \sim F_Y(y)$.

Statistical problem

Modeling the relationship between scores to make them comparable.

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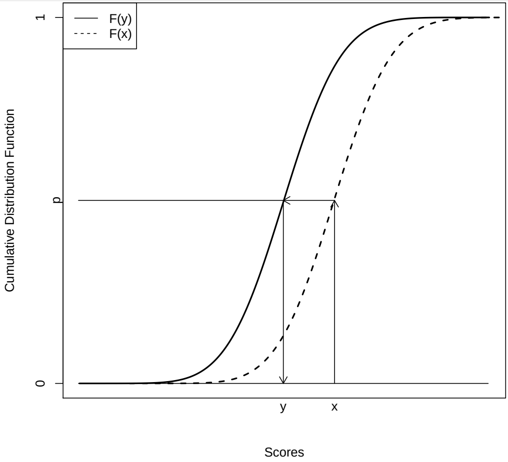
General formulation: equating transformation

Definition

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sample spaces. A function $\varphi : \mathcal{X} \mapsto \mathcal{Y}$ will be called an *equating transformation*.

- The equating transformation maps the scores on the scale of one test form into the scale of the other.
- The equating transformation is to be estimated by an estimator φ_n based on samples $x_1, \ldots, x_{n_x} \sim F_X$ and $y_1, \ldots, y_{n_y} \sim F_Y$, respectively.

Equipercentile transformation



- $y = \varphi(x) = F_Y^{-1}(F_X(x))$

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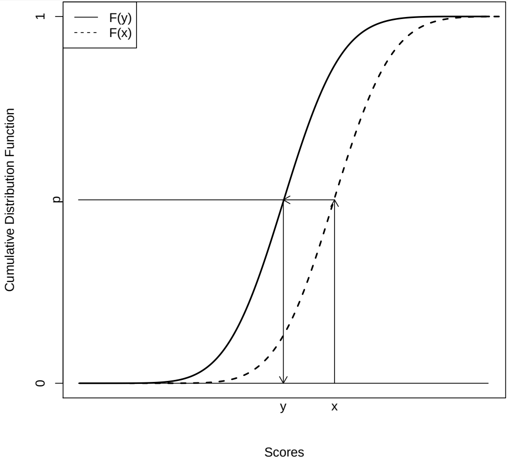

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NEAT equating design

Population	Sample	X	Y	A
P	1	✓		✓
Q	2		✓	✓

$$T = w_P P + w_Q Q$$

$$\begin{aligned} f_{XT}(x) &= w_P f_{XP}(x) + w_Q f_{XQ}(x) \\ f_{YT}(y) &= w_P f_{YP}(y) + w_Q f_{YQ}(y). \end{aligned}$$

- Additional assumptions are needed to estimate $f_{XT}(x)$ and $f_{YT}(y)$

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NEAT equating design

- Most common assumption,

$$f_{XP}(x \mid a) = f_{XQ}(x \mid a) \quad \text{and} \quad f_{YP}(y \mid a) = f_{YQ}(y \mid a)$$

- Using these assumptions we get

$$f_{XT}(x) = w_P f_{XP}(x) + w_Q \sum_a f_{XP}(x) f_{AQ}(a)$$

$$f_{YT}(y) = w_P \sum_a f_{YQ}(y) f_{AP}(a) + w_Q f_{YQ}(y)$$

and from here

$$\varphi_T(x) = F_{YT}^{-1}(F_{XT}(x))$$

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Conditional score distributions with no assumptions

- Let Z denote the population group so that

$$Z = \begin{cases} 1, & \text{if test taker is administered X;} \\ 0, & \text{if test taker is administered Y.} \end{cases}$$

- Then, by the Law of Total Probability (LTP)

$$\begin{aligned} P(X \leq x \mid A) = & P(X \leq x \mid A, Z = 1) P(Z = 1 \mid A) + \\ & P(X \leq x \mid A, Z = 0) P(Z = 0 \mid A) \end{aligned} \quad (1)$$

- $P(X \leq x \mid A)$ is not identified.

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Partially identified probability distributions

- From (1) and the fact that $P(X \leq x \mid A, Z = 0)$ is bounded between 0 and 1, it follows that

$$L_x \leq P(X \leq x \mid A) \leq U_x \tag{2}$$

$L_x = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A)$
$U_x = P(X \leq x \mid A, Z = 1)P(Z = 1 \mid A) + P(Z = 0 \mid A)$

- Analogously for Y we have

$$L_y \leq P(Y \leq y \mid A) \leq U_y \tag{3}$$

$L_y = P(Y \leq y \mid A, Z = 0)P(Z = 0 \mid A)$
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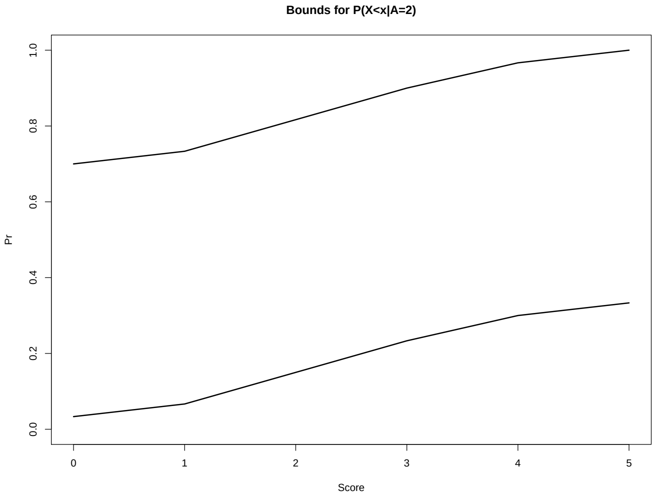
Graphical illustration



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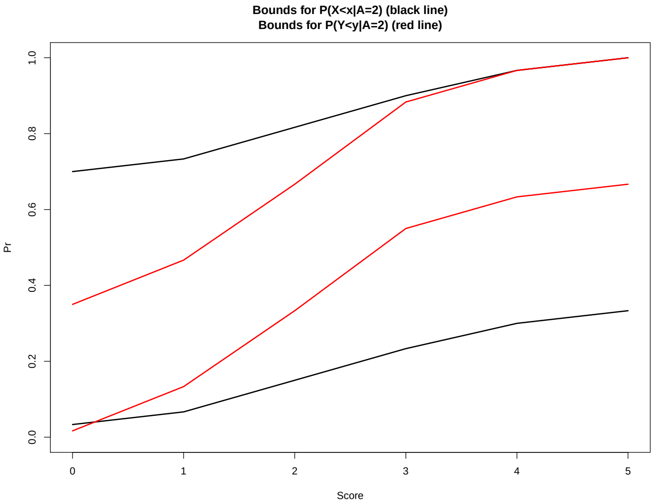

Target distributions with no assumptions

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  becomes

$$P(X \leq x) = P(X \leq x \mid Z = 1)P(Z = 1) + P(X \leq x \mid Z = 0)P(Z = 0)$$

  or

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Partially identified target distributions

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Bounds illustrations

Score	F_{XT}	$[L_x, U_x]$	F_{YT}	$[L_y, U_y]$
0	0.100	[0.050 ; 0.550]	0.105	[0.040 ; 0.540]
1	0.250	[0.125 ; 0.625]	0.320	[0.140 ; 0.640]
2	0.500	[0.250 ; 0.750]	0.530	[0.250 ; 0.750]
3	0.750	[0.375 ; 0.875]	0.755	[0.375 ; 0.875]
4	0.900	[0.450 ; 0.950]	0.900	[0.450 ; 0.950]
5	1.000	[0.500 ; 1.000]	1.000	[0.500 ; 1.000]

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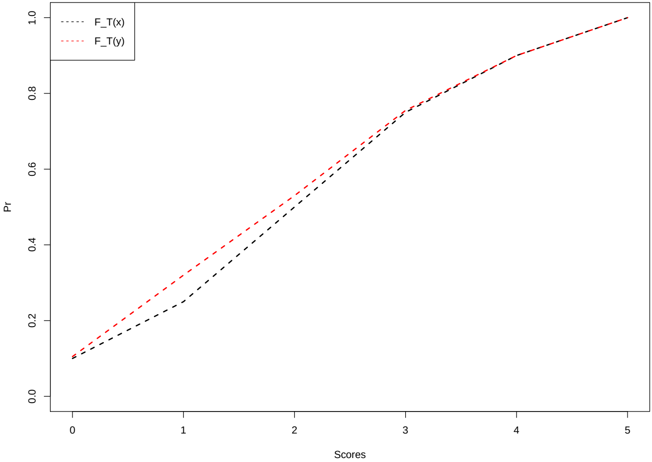

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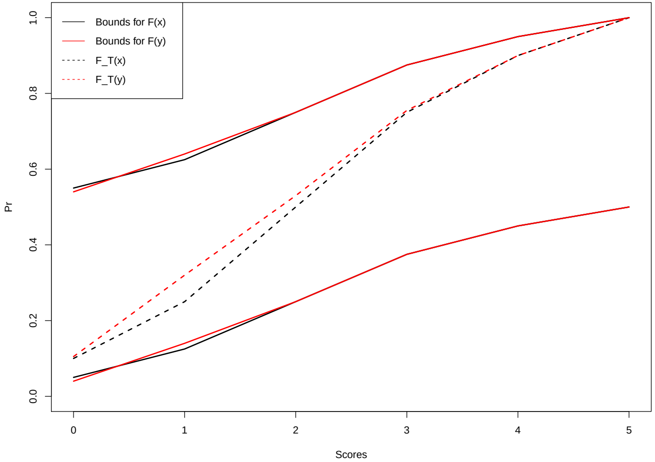

Bounded quantiles

- Let $\alpha \in (0,1)$ and $q_\alpha(X) \doteq \inf\{t : P(X \leq t) > \alpha\}$. Define the following quantiles:

$$
\begin{aligned}
r_\alpha(X) \doteq & \inf\{t : P(X \leq t \mid Z = 1)P(Z = 1) + P(Z = 0) > \alpha\} \\
= & \inf\left\{t : P(X \leq t \mid Z = 1) > \frac{\alpha - P(Z = 0)}{P(Z = 1)}\right\} \\
= & q_{\alpha^*}(X \mid Z = 1)
\end{aligned}
$$

and

$$
\begin{aligned}
s_\alpha(X) \doteq & \inf\{t : P(X \leq t \mid Z = 1)P(Z = 1) > \alpha\} \\
= & \inf\left\{t : P(X \leq t \mid Z = 1) > \frac{\alpha}{P(Z = 1)}\right\} \\
= & q_{\alpha'}(X \mid Z = 1)
\end{aligned}
$$

Bounded quantiles

- In the NEAT design, the quantiles of the partially identified probabilities $P(X \leq t)$ and $P(Y \leq t)$ are also partially identified by the following intervals:

$$\begin{array}{rl} \text{(i)} & r_\alpha(X) \leq q_\alpha(X) \leq s_\alpha(X); \\ \text{(ii)} & r_\alpha(Y) \leq q_\alpha(Y) \leq s_\alpha(Y). \end{array} \tag{5}$$

Bounded quantile equating

- Main idea

α	$[r_\alpha(X), s_\alpha(X)]$	$[r_\alpha(Y), s_\alpha(Y)]$
$\vdots$	$\vdots$	$\vdots$
0.1	[1 ; 3]	[2 ; 4]
0.2	[2 ; 4]	[3 ; 5]
$\vdots$	$\vdots$	$\vdots$

Informative bounded quantiles

- The lower and the upper bounds are not always informative:

$$0 \leq \frac{\alpha - P(Z=0)}{P(Z=1)} \leq 1, \quad \text{for all } \alpha \in [P(Z=0), 1];$$

$$0 \leq \frac{\alpha}{P(Z=1)} \leq 1, \quad \text{for all } \alpha \in [0, P(Z=1)].$$

Bounded equipercentile equating

- Given that

$$F_{X_P}(t) \cdot w \leq P(X \leq t) \leq F_{X_P}(t) \cdot w + (1 - w)$$

then

$$F_Y^{-1}(F_{X_P}(t) \cdot w) \leq \varphi(t) \leq F_Y^{-1}(F_{X_P}(t) \cdot w + (1 - w))$$

moreover

$$u_Y(\alpha) = \inf\{t : F_Y(t) > F_{Y_Q}(t) \cdot (1 - w) > \alpha\}$$

and

$$l_Y(\alpha) = \inf\{t : F_{Y_Q}(t) \cdot (1 - w) + w > F_Y(t) > \alpha\}$$

- Data from Kolen and Brennan (2014). Two 36-items test forms. Form X was administered to 1,655 examinees and form Y was administered to 1,638 examinees. Also, 12 out of the 36 items are common between both test forms (items 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, and 36).
- $w_P = 1$ and $w_Q = 0$

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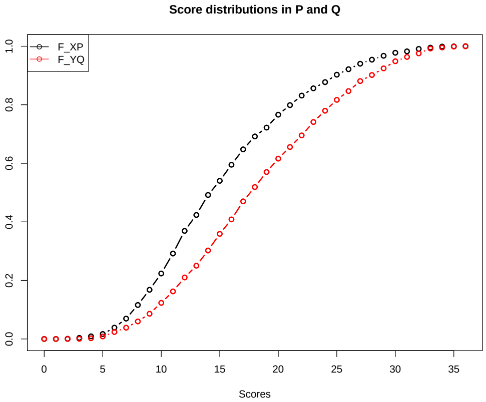

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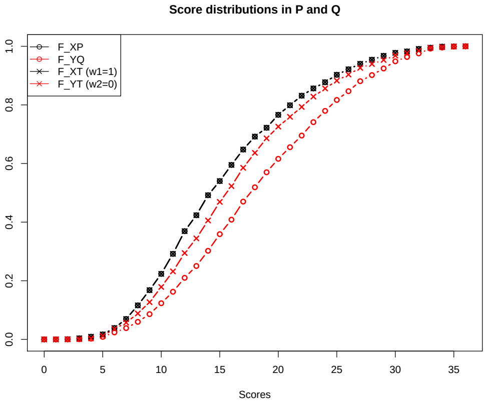

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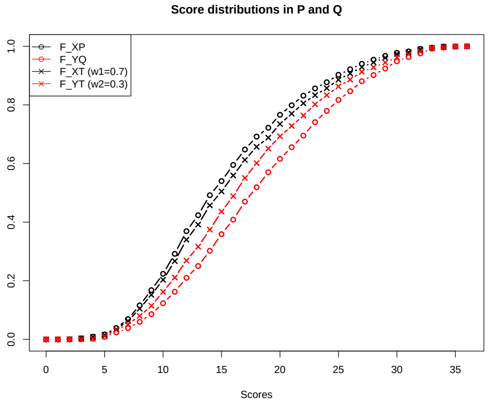

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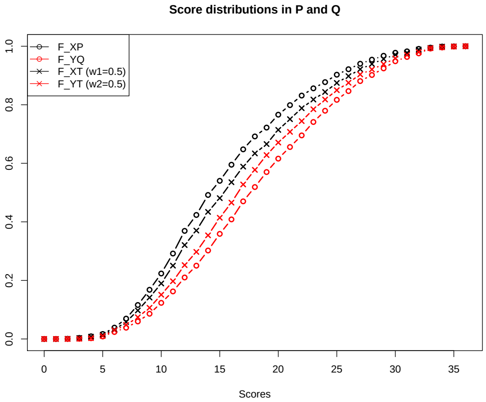

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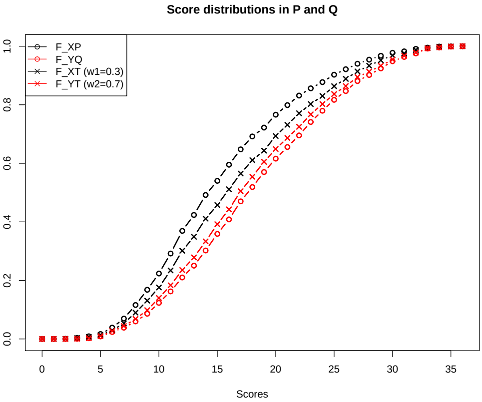

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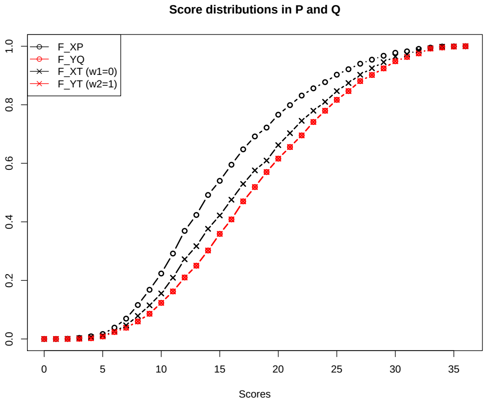

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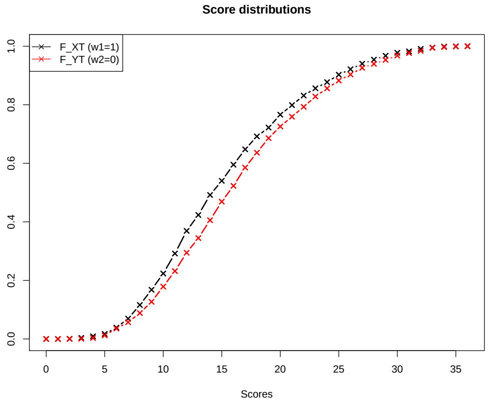

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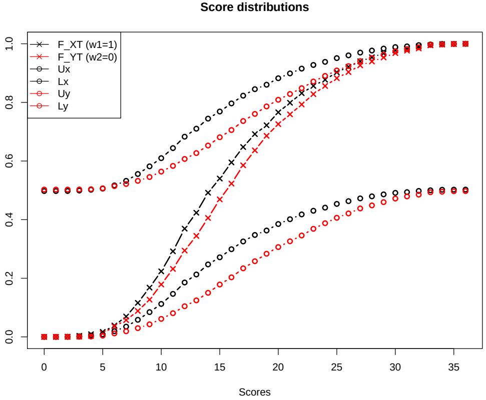

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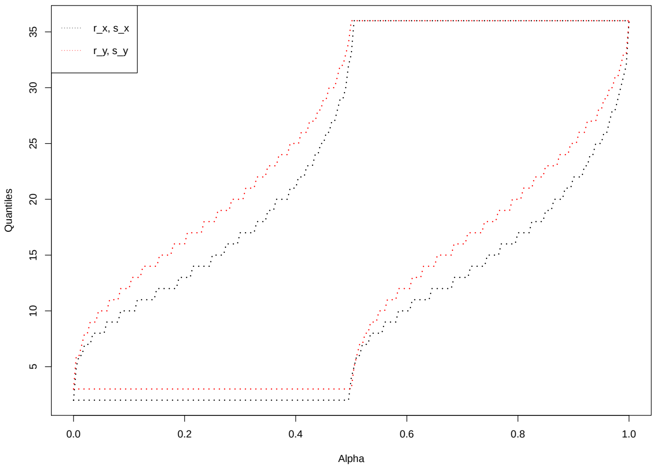

Summary and discussion

- The equality of conditional distributions assumption is actually an *identification restriction*.
- We offer an alternative to the conditional distributions restriction $\rightarrow$ Partially identified probability distributions
- The derived bounds show that there is huge uncertainty about the probability distributions that are to be used for equating

Future work

- Equiquantile vs equipercentile: which one?
- Incorporate additional information in order to obtain tighter bounds.

Thank you for your attention!

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This research is supported by Grant Fondecyt 1150233